PROOF OF THE KURLBERG–RUDNICK RATE CONJECTURE

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To Joseph Bernstein for his 60's birthday with admiration

Abstract

In this paper we present a proof of the *Hecke quantum unique ergodicity rate conjecture* for the Hannay-Berry model. A model of quantum mechanics on the two-dimensional torus. This conjecture was stated in Z. Rudnick's lectures at MSRI, Berkeley 1999, and ECM, Barcelona 2000.

0   Introduction

0.1   Hannay-Berry model

In the paper "*Quantization of linear maps on the torus - Fresnel diffraction by a periodic grating*", published in 1980 [HB], the physicists J. Hannay and Sir M.V. Berry explore a model for quantum mechanics on the two-dimensional symplectic torus $(\mathbb{T}, \omega)$. Hannay and Berry suggested to quantize simultaneously the functions on the torus and the linear symplectic group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

0.2   Quantum chaos

One of their main motivations was to study the phenomenon of quantum chaos [B1, B2, R2, S] in this model. More precisely, they considered an ergodic discrete dynamical system on the torus, which is generated by a hyperbolic automorphism $A \in \mathrm{SL}_2(\mathbb{Z})$. Quantizing the system, we replace: the classical phase space $(\mathbb{T}, \omega)$ by a *finite dimensional* Hilbert space $\mathcal{H}_\hbar$, classical observables, i.e., functions $f \in C^\infty(\mathbb{T})$, by operators $\pi_\hbar(f) \in \mathrm{End}(\mathcal{H}_\hbar)$, and classical symmetries by a unitary representation $\rho_\hbar : \mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{U}(\mathcal{H}_\hbar)$. A fundamental meta-question in the area of quantum chaos is to *understand* the ergodic properties of the quantum system $\rho_\hbar(A)$, at least in the semi-classical limit as $\hbar \to 0$.

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0.3 Schnirelman's theorem

analogous with the case of the Schrödinger equation, consider the following eigenstate problem:

$$\rho_{\hbar}(A)\Psi = \lambda\Psi, \quad \Psi \in \mathcal{H}_{\hbar}. \tag{0.3.1}$$

A fundamental result, valid for a wide class of quantum systems which are associated to ergodic classical dynamics, is Schnirelman's theorem [Sc], asserting that in the semi-classical limit "almost all" eigenstates becomes equidistributed in an appropriate sense. A variant of Schnirelman's theorem also holds in our situation [BD]. More precisely, we have that in the semi-classical limit $\hbar \longrightarrow 0$, for "almost all" eigenstates Ψ of the operator $\rho_{\hbar}(A)$, the corresponding *Wigner distribution* $\langle\Psi|\pi_{\hbar}(\cdot)\Psi\rangle : C^{\infty}(\mathbb{T}) \longrightarrow \mathbb{C}$ approaches the phase space average $\int_{\mathbb{T}} \cdot |\omega|$. In this respect, it seems natural to ask whether there exists exceptional sequences of eigenstates? Namely, eigenstates that do not obey the Schnirelman's rule ("scarred" eigenstates). It was predicted by Berry [B1, B2], that "scarring" phenomenon is not expected to be seen for quantum systems associated with "generic" chaotic classical dynamics. However, in our situation, the operator $\rho_{\hbar}(A)$ is not generic, and exceptional eigenstates were constructed. Indeed, it was observed numerically, and then confirmed mathematically in [FND], that certain $\rho_{\hbar}(A)$-eigenstates might localize. For example, in that paper, a sequence of eigenstates Ψ was constructed, for which the corresponding Wigner distribution approaches the measure $\frac{1}{2}\delta_0 + \frac{1}{2}|\omega|$ on $\mathbb{T}$.

0.4 Hecke quantum unique ergodicity

A quantum system that obeys the Schnirelman's rule is also called quantum ergodic. Can one impose some natural conditions on the eigenstates (0.3.1) so that no exceptional eigenstates will appear? Namely, *Quantum Unique Ergodicity* will hold. This question was addressed in a paper by Kurlberg and Rudnick [KR1]. In this paper they formulated a rigorous notion of Hecke quantum unique ergodicity for the case $\hbar = \frac{1}{p}$. The following is a brief description of that work. The basic observation is that the degeneracies of the operator $\rho_{\hbar}(A)$ are coupled with the existence of symmetries. There exists a commutative group of operators that commutes with $\rho_{\hbar}(A)$. In more detail, the representation $\rho_{\hbar}$ factors through the quotient group $\Gamma_p \simeq \mathrm{SL}_2(\mathbb{F}_p)$. We denote by $\mathrm{T}_A \subset \Gamma_p$ the centralizer of the element A, now considered as an element of the quotient group Γ_p. The group T_A is called (cf. [KR1]) the *Hecke torus* corresponding to the element A. The Hecke torus acts semisimply on $\mathcal{H}_{\hbar}$. Therefore, we have a decomposition:

$$\mathcal{H}_{\hbar} = \bigoplus_{\chi:\mathrm{T}_A \longrightarrow \mathbb{C}^*} \mathcal{H}_{\chi},$$

where $\mathcal{H}_{\chi}$ is the Hecke eigenspace corresponding to the character χ. Consider a unit eigenstate $\Psi \in \mathcal{H}_{\chi}$ and the corresponding Wigner distribution $\mathcal{W}_{\chi} : C^{\infty}(\mathbb{T}) \longrightarrow \mathbb{C}$, defined by the formula $\mathcal{W}_{\chi}(f) = \langle\Psi|\pi_{\hbar}(f)\Psi\rangle$. The main statement in [KR1] asserts about an explicit bound of the semi-classical asymptotic of $\mathcal{W}_{\chi}(f)$:

$$\left|\mathcal{W}_{\chi}(f) - \int_{\mathbb{T}} f|\omega|\right| \leq \frac{C_f}{p^{1/4}},$$

where C_f is a constant that depends only on the function f. In Rudnick's lectures at MSRI, Berkeley 1999 [R1], and ECM, Barcelona 2000 [R2], he conjectured that a stronger bound should hold true, that is,:

Conjecture 0.1 (Rate Conjecture) *The following bound holds:*

$$\left| \mathcal{W}_\chi(f) - \int_\mathbb{T} f |\omega| \right| \leq \frac{C_f}{p^{1/2}} \, .$$

The basic *clues* suggesting the validity of this stronger bound come from two main sources. The first source is *computer* simulations [Ku] accomplished over the years to give extremely precise bounds for considerably large values of p. A more mathematical argument is based on the fact that for special values of p, in which the Hecke torus *splits*, namely, $\mathrm{T}_A \simeq \mathbb{F}_p^*$, one is able to compute explicitly the eigenstate $\Psi \in \mathcal{H}_\chi$ and as a consequence to give an explicit *formula* for the Wigner distribution [KR2, DGI]. More precisely, in case $\xi \in \mathbb{T}^\vee$, i.e., a character, the distribution $\mathcal{W}_\chi(\xi)$ turns out to be equal to the exponential sum:

$$\frac{1}{p} \sum_{a \in \mathbb{F}_p^*} \psi\left(\frac{a+1}{a-1}\right) \sigma(a)\chi(a),$$

where σ denotes the Legendre character, and ψ is a non-trivial additive character of $\mathbb{F}_p$. This sum is very much similar to the Kloosterman sum and the classical Weil bound [W1] yields the result.

In this paper, a proof for the rate conjecture is presented, treating both cases of split and inert (*non-split*) tori in a uniform manner. A fundamental idea in our approach concerns a non-trivial relation between two seemingly different dynamical systems. One which is attached to a split ("non-compact") torus and the other which is attached to a non-split ("compact") torus. This relation is geometric in nature and can be formally described in the framework of algebraic geometry.

0.5   Geometric approach

The basic observation to be made is that the theory of quantum mechanics on the torus, in case $\hbar = \frac{1}{p}$, can be equivalently recast in the language of representation theory of finite groups in characteristic p. We will endeavor to give a more precise explanation of this matter. Consider the quotient $\mathbb{F}_p$-vector space $\mathrm{V} = \mathbb{T}^\vee/p\mathbb{T}^\vee$, where $\mathbb{T}^\vee$ is the lattice of characters on $\mathbb{T}$. We denote by $\mathrm{H} = \mathrm{H}(\mathrm{V})$ the Heisenberg group. The group $\Gamma_p \simeq \mathrm{SL}_2(\mathbb{F}_p)$ is naturally identified with the group of linear symplectomorphisms of V. We have an action of $\mathrm{SL}_2(\mathbb{F}_p)$ on H. The Stone-von Neumann theorem states that there exists a unique irreducible representation $\pi : \mathrm{H} \longrightarrow \mathrm{GL}(\mathcal{H})$, with the non-trivial central character ψ, for which its isomorphism class is fixed by $\mathrm{SL}_2(\mathbb{F}_p)$. This is equivalent to saying that $\mathcal{H}$ is equipped with a compatible projective representation $\rho : \mathrm{SL}_2(\mathbb{F}_p) \longrightarrow \mathrm{PGL}(\mathcal{H})$. Noting that H and $\mathrm{SL}_2(\mathbb{F}_p)$ are the sets of rational

points of corresponding algebraic groups, it is natural to *ask* whether there exists an algebra-geometric object that underlies the pair (π, ρ)?. The answer to this question is *positive*. The construction is proposed in an unpublished letter of Deligne to Kazhdan [D1]. In one sentence, the content of this letter is a construction of *Representation Sheaves* $\mathcal{K}_\pi$ and $\mathcal{K}_\rho$ on the algebraic varieties $\mathbf{H}$ and $\mathbf{SL_2}$ respectively. One obtains, as a consequence, the following general principle:

(*) Motivic principle: *All quantum mechanical quantities in the Hannay-Berry model are motivic in nature.*

By this we mean that every quantum-mechanical quantity $\mathcal{Q}$, is associated with a vector space $V_\mathcal{Q}$ endowed with a Frobenius action $\mathrm{Fr} : V_\mathcal{Q} \longrightarrow V_\mathcal{Q}$, so that:

$$\mathcal{Q} = \mathrm{Tr}(\mathrm{Fr}_{|V_\mathcal{Q}}).$$

The *main contribution* of this paper is to implement this principle. In particular, we show that there exists a two-dimensional vector space V_χ, endowed with an action $\mathrm{Fr} : V_\chi \longrightarrow V_\chi$, so that:

$$\mathcal{W}_\chi(\xi) = \mathrm{Tr}(\mathrm{Fr}_{|V_\chi}).$$

This, combined with a bound on the modulus of the eigenvalues of Frobenius, i.e.,

$$\left| \mathrm{e.v}(\mathrm{Fr}_{|V_\chi}) \right| \leq \frac{1}{p^{1/2}},$$

completes the proof of the rate conjecture.

0.6 Remarks

There are several remarks that we would like to make at this point:

Remark 1: Discreteness principle. "Every" quantity $\mathcal{Q}$ that appears in the Hannay-Berry model admits discrete spectrum in the following arithmetic sense: $\mathcal{Q}$ can take only values which are finite linear combinations of terms with absolute value of the form $p^{i/2}$ for $i \in \mathbb{Z}$. This is a consequence of the motivic principle (*) and Deligne's weight theory [D2]. This puts some restrictions on the possible values of the modulus $|\mathcal{Q}|$. We believe that this principle can be effectively used in various situations in order to derive strong bounds out of weaker bounds. A *striking example* would be a possible alternative trivial "proof" for the bound $|\mathcal{W}_\chi(\xi)| \leq \frac{C_\xi}{p^{1/2}}$:

$$|\mathcal{W}_\chi(\xi)| \leq \frac{C_\xi}{p^{1/4}} \ \Rightarrow \ |\mathcal{W}_\chi(\xi)| \leq \frac{C_\xi}{p^{1/2}}.$$

Kurlberg and Rudnick proved in their paper [KR1] the weak bound $|\mathcal{W}_\chi(\xi)| \leq \frac{C_\xi}{p^{1/4}}$. This strongly indicates that the stronger bound $|\mathcal{W}_\chi(\xi)| \leq \frac{C_\xi}{p^{1/2}}$ is valid.

Remark 2: Higher dimensional exponential sums. Proving the bound $|\mathcal{W}_\chi(f)| \leq \frac{C_f}{\sqrt{p}}$ can be equivalently stated as bounding by $\frac{C_f}{\sqrt{p}}$ the spectral radius of the operator $\mathbf{Av}_{T_A}(f) = \frac{1}{|T_A|} \sum\limits_{B \in T_A} \rho_\hbar(B) \pi_\hbar(f) \rho_\hbar(B^{-1})$. This implies a bound on the L_N norms, for every $N \in \mathbb{Z}^+$:

$$\|\mathbf{Av}_{T_A}(f)\|_N \leq \frac{C_f}{p^N}. \tag{0.6.1}$$

In particular, for $0 \neq f = \xi \in \mathbb{T}^\vee$ one can compute explicitly the left hand side of (0.6.1) and obtain:

$$\|\mathbf{Av}_{T_A}(\xi)\|_N = \text{Tr}(|\mathbf{Av}_{T_A}(\xi)|^N) = \frac{1}{|T_A|^{2N}} \sum_{(x_1,\ldots,x_{2N}) \in X} \psi(\sum_{i<j} \omega(x_i, x_j)),$$

where $X = \{(x_1, \ldots, x_{2N}) | \; x_i \in \mathcal{O}_\xi, \; \sum x_i = 0\}$ and $\mathcal{O}_\xi = T_A \cdot \xi \subset V$ denotes the orbit of ξ under the action of T_A. Therefore, referring to (0.6.1) we obtained a *non-trivial* bound for a higher dimensional exponential sum. It would be interesting to know whether there exists an independent proof for this bound and whether this representation theoretic approach can be used to prove optimal bounds for other interesting higher dimensional exponential sums.

0.7 Sato-Tate conjecture

The next level of the theory is to understand the *complete statistics* of the Hecke-Wigner distributions for different Hecke states. More precisely, let us fix a character $\xi \in \mathbb{T}^\vee$. For every character $\chi : T_A \longrightarrow \mathbb{C}^*$ we consider the normalized value $\tilde{\mathcal{W}}_\chi(\xi) = \frac{1}{\sqrt{p}} \mathcal{W}_\chi(\xi)$, which lies in the interval $[-2, 2]$. Now, running over all multiplicative characters we define the following atomic measure on the interval $[-2, 2]$:

$$\mu_p = \frac{1}{|T_A|} \sum_\chi \delta_{\tilde{\mathcal{W}}_\chi(\xi)}.$$

One would like to describe the limit measure (if it exists!). This is the content of another conjecture of Kurlberg and Rudnick [KR2]:

Conjecture (Sato-Tate Conjecture). The following limit exists:

$$\lim_{p \to \infty} \mu_p = \mu_{ST},$$

where μ_{ST} is the push-forward of the Haar measure on $SU(2)$ to the interval $[-2, 2]$ through the map $g \mapsto \text{Tr}(g)$.

We hope that using the methodology described in this paper one will be able to gain some progress in proving this conjecture.

Remark. Note that the family $\{\tilde{\mathcal{W}}_\chi(\xi)\}_{\chi \in T_A^\vee}$ runs over a non-algebraic space of parameters. Hence Deligne's equidistribution theory (cf. Weil II [D2]) can not be applied directly in order to solve the Sato-Tate Conjecture.

0.8 Results

1. **Kurlberg-Rudnick conjecture.** The main result of this paper is Theorem 3.1, which is the *proof* of the Kurlberg-Rudnick rate conjecture (Conjecture 0.1) on the asymptotic behavior of the Hecke-Wigner distributions.

2. **Weil representation.** We introduce two new constructions of the Weil representation over finite fields.

   (a) The *first* construction is stated in Theorem 2.2, and is based on the Rieffel *quantum torus* $\mathcal{A}_\hbar$, for $\hbar = \frac{1}{p}$. This approach is essentially equivalent to the classical approach [Ge, H, Kl, W2] that uses the representation theory of the Heisenberg group in characteristic p. The fundamental difference is that the quantum torus is well defined for every value of the parameter $\hbar$.

   (b) **Canonical Hilbert space (Kazhdan's question).** The *second* construction uses the *"method of Canonical Hilbert Space"* (see Appendix A). This approach is based on the following statement:

   Proposition (Canonical Hilbert Space). Let (V, ω) be a two-dimensional symplectic vector space over the finite field $\mathbb{F}_q$. There exists a canonical Hilbert space $\mathcal{H}_V$ attached to (V, ω).

   An immediate consequence of this proposition is that all symmetries of (V, ω) automatically act on $\mathcal{H}_V$. In particular, we obtain a *linear* representation of the group $\mathrm{Sp} = \mathrm{Sp}(V, \omega)$ on $\mathcal{H}_V$. This approach has higher dimensional generalization, for the case where V is of dimension $2n$. This generalization will be published by the authors elsewhere.

   Remark. Note the main difference of our construction from the classical approach due to Weil (cf. [W2]). The *classical construction* proceeds in two stages. Firstly, one obtains a projective representation of Sp, and secondly, using general arguments about the group Sp one proves the existence of a linearization. A consequence of *our approach* is that there exists a *distinguished* linear representation, and its existence is not related to any group theoretic property of Sp. We would like to mention that this approach answers, in the case of the two-dimensional Heisenberg group, *a question* of David Kazhdan [Ka] dealing with the existence of Canonical Hilbert Spaces for co-adjoint orbits of general unipotent groups. The main motive behind our construction is the notion of *oriented Lagrangian subspace*. This idea was suggested to us by Joseph Bernstein [B].

3. **Deligne's Weil representation sheaf.** We include for the sake of completeness (see Appendix A.4) a formal presentation of *Deligne's letter to Kazhdan* [D1], that places the Weil representation on a complete algebro-geometric ground. As far as we know, the content of this letter was never published. This construction plays a *central rule* in the proof of the Kurlberg-Rudnick conjecture.

0.9 Structure of the paper

The paper is naturally separated into four parts:

Part I. Consists of sections 1, 2 and 3. In section 1 we discuss classical mechanics on the torus. In section 2 we discuss quantum mechanics á-la Hannay and Berry, using the Rieffel quantum torus model. In section 3 we *formulate* the quantum unique ergodicity theorem, i.e., Theorem 3.1. This part of the paper is self-contained and consists of mainly linear algebraic considerations.

Part II. Section 4. This is the main part of the paper, consisting of the *proof* of the Kurlberg-Rudnick conjecture. The proof is given in two stages. The first stage consists of mainly linear algebraic manipulations to obtain a more transparent formulation of the statement, resulting in Theorem 4.2. In the second stage we venture into algebraic geometry. All linear algebraic constructions are replaced by sheaf theoretic objects, concluding with the *Geometrization Theorem*, i.e., Theorem 4.4. Next, the statement of Theorem 4.2 is reduced to a geometric statement, the *Vanishing Lemma*, i.e., Lemma 4.6. The remainder of the section is devoted to the proof of Lemma 4.6. For the convenience of the reader we include a large body of intuitive explanations for all the constructions involved. In particular, we devote some space explaining the Grothendieck *Sheaf to Function Correspondence* procedure which is the basic bridge connecting Parts I and II.

Part III: Appendix A. In section A.1 we describe the *method of canonical Hilbert space*. In section A.2 we describe the Weil representation in this manifestation. In section A.3 we relate the invariant construction to the more classical constructions, supplying explicit formulas that will be used later. In section A.4 we give a formal presentation of *Deligne's letter to Kazhdan* [D1]. The main statement of this section is Theorem A.5, in which the *Weil representation sheaf* $\mathcal{K}$ is introduced. We include in our presentation only the parts of that letter which are most relevant to our needs. In particular, we consider only the two-dimensional case of this letter. In section A.5 we supply proofs for all technical lemmas and propositions appearing in the previous sections of the Appendix.

Part IV: Appendix B. In this Appendix we supply the proofs for all statements appearing in Part I and Part II. In particular, we give the proof of Theorem 4.4 which essentially consists of taking the *Trace* of Deligne's *Weil representation sheaf $\mathcal{K}$*.

Acknowledgments

1 Classical Torus

Let $(\mathbb{T}, \omega)$ be the two-dimensional symplectic torus. Together with its linear symplectomorphisms $\Gamma \simeq \mathrm{SL}_2(\mathbb{Z})$ it serves as a simple model of classical mechanics (a compact version of the phase space of the harmonic oscillator). More precisely, let $\mathbb{T} = \mathrm{W}/\Lambda$ where W is a two-dimensional real vector space, i.e., $\mathrm{W} \simeq \mathbb{R}^2$ and Λ is a rank two lattice in W, i.e., $\Lambda \simeq \mathbb{Z}^2$. We obtain the symplectic form on $\mathbb{T}$ by taking a non-degenerate symplectic form on W:

$$\omega : \mathrm{W} \times \mathrm{W} \longrightarrow \mathbb{R}.$$

We require ω to be integral, namely, $\omega : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ and normalized, i.e., $\mathrm{Vol}(\mathbb{T}) = 1$.

Let $\mathrm{Sp}(\mathrm{W}, \omega)$ be the group of linear symplectomorphisms, i.e., $\mathrm{Sp}(\mathrm{W}, \omega) \simeq \mathrm{SL}_2(\mathbb{R})$. Consider the subgroup $\Gamma \subset \mathrm{Sp}(\mathrm{W}, \omega)$ of elements that preserve the lattice Λ, i.e., $\Gamma(\Lambda) \subseteq \Lambda$. Then $\Gamma \simeq \mathrm{SL}_2(\mathbb{Z})$. The subgroup Γ is the group of linear symplectomorphisms of $\mathbb{T}$.
We denote by $\Lambda^* \subseteq \mathrm{W}^*$ the dual lattice $\Lambda^* = \{\xi \in \mathrm{W}^* | \ \xi(\Lambda) \subset \mathbb{Z}\}$. The lattice Λ^* is identified with the lattice of characters of $\mathbb{T}$ by the following map:

$$\xi \in \Lambda^* \longmapsto e^{2\pi i <\xi, \cdot>} \in \mathbb{T}^\vee,$$

where $\mathbb{T}^\vee = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$.

1.1 Classical mechanical system

We consider a very simple discrete mechanical system. An hyperbolic element $A \in \Gamma$, i.e., $|\mathrm{Tr}(A)| > 2$, generates an ergodic discrete dynamical system. The *Birkhoff's Ergodic Theorem* states that:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(A^k x) = \int_{\mathbb{T}} f|\omega|,$$

for every $f \in \mathcal{S}(\mathbb{T})$ and for almost every point $x \in \mathbb{T}$. Here $\mathcal{S}(\mathbb{T})$ stands for a good class of functions, for example trigonometric polynomials or smooth functions.

We fix an hyperbolic element $A \in \Gamma$ for the remainder of the paper.

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2 Quantization of the Torus

Quantization is one of the big mysteries of modern mathematics, indeed it is not clear at all what is the precise structure which underlies quantization in general. Although physicists have been using quantization for almost a century, for mathematicians the concept remains all-together unclear. Yet, in specific cases, there are certain formal models for quantization that are well justified mathematically. The case of the symplectic torus is one of these cases. Before we employ the formal model, it is worthwhile to discuss the general phenomenological principles of quantization which are surely common for all models.

Let us start with a model of classical mechanics, namely, a symplectic manifold, serving as a classical phase space. In our case this manifold is the symplectic torus $\mathbb{T}$. Principally, quantization is a protocol by which one associates a quantum "phase" space $\mathcal{H}$ to the classical phase space $\mathbb{T}$, where $\mathcal{H}$ is a Hilbert space. In addition, the protocol gives a rule by which one associates to every classical observable, namely a function $f \in \mathcal{S}(\mathbb{T})$, a quantum observable $\mathrm{Op}(f) : \mathcal{H} \longrightarrow \mathcal{H}$, an operator on the Hilbert space. This rule should send a real function into a self adjoint operator.

To be more precise, quantization should be considered not as a single protocol, but as a one parameter family of protocols, parameterized by $\hbar$, the Planck constant. For every fixed value of the parameter $\hbar$ there is a protocol which associates to $\mathbb{T}$ a Hilbert space $\mathcal{H}_\hbar$ and for every function $f \in \mathcal{S}(\mathbb{T})$ an operator $\mathrm{Op}_\hbar(f) : \mathcal{H}_\hbar \longrightarrow \mathcal{H}_\hbar$. Again the association rule should send real functions to self adjoint operators.

Accepting the general principles of quantization, one searches for a formal model by which to quantize, that is a mathematical model which will manufacture a family of Hilbert spaces $\mathcal{H}_\hbar$ and association rules $\mathcal{S}(\mathbb{T}) \rightsquigarrow \mathrm{End}(\mathcal{H}_\hbar)$. In this work we employ a model of quantization called the *Weyl quantization model*.

2.1 The Weyl quantization model

The Weyl quantization model works as follows. Let $\mathcal{A}_\hbar$ be a one parameter deformation of the algebra $\mathcal{A}$ of trigonometric polynomials on the torus. This algebra is known in the literature as the Rieffel torus [Ri]. The algebra $\mathcal{A}_\hbar$ is constructed by taking the free algebra over $\mathbb{C}$ generated by the symbols $\{s(\xi) \mid \xi \in \Lambda^*\}$ and quotient out by the relation $s(\xi + \eta) = e^{\pi i \hbar \omega(\xi, \eta)} s(\xi) s(\eta)$. Here ω is the form on W^* induced by the original form ω on W. We point out two facts about the algebra $\mathcal{A}_\hbar$. First, when substituting $\hbar = 0$ one gets the group algebra of Λ^*, which is exactly equal to the algebra of trigonometric polynomials on the torus. Second, the algebra $\mathcal{A}_\hbar$ contains as a standard basis the lattice Λ^*:

$$s : \Lambda^* \longrightarrow \mathcal{A}_\hbar.$$

Therefore, one can identify the algebras $\mathcal{A}_\hbar \simeq \mathcal{A}$ as vector spaces. Therefore, every function $f \in \mathcal{A}$ can be viewed as an element of $\mathcal{A}_\hbar$.

For a fixed $\hbar$ a representation $\pi_\hbar : \mathcal{A}_\hbar \longrightarrow \mathrm{End}(\mathcal{H}_\hbar)$ serves as a quantization protocol, that is, for every function $f \in \mathcal{A}$ one has:

$$f \in \mathcal{A} \simeq \mathcal{A}_\hbar \longmapsto \pi_\hbar(f) \in \mathrm{End}(\mathcal{H}_\hbar).$$

An equivalent way of saying this is:

$$f \longmapsto \sum_{\xi \in \Lambda^*} a_\xi \pi_\hbar(\xi),$$

where $f = \sum_{\xi \in \Lambda^*} a_\xi \cdot \xi$ is the Fourier expansion of f.

To summarize: every family of representations $\pi_\hbar : \mathcal{A}_\hbar \longrightarrow \mathrm{End}(\mathcal{H}_\hbar)$ gives us a complete quantization protocol. Yet, a serious question now arises, namely what representations to choose? Is there a correct choice of representations, both mathematically, but also perhaps physically? A possible restriction on the choice is to choose an irreducible representation. Yet, some ambiguity still remains because there are several irreducible classes for specific values of $\hbar$.

We present here a partial solution to this problem in the case where the parameter $\hbar$ is restricted to take only rational values [GH1]. Even more particularly, we take $\hbar$ to be of the form $\hbar = \frac{1}{p}$ where p is an odd prime number. Before any formal discussion one should recall that our classical object is the symplectic torus $\mathbb{T}$ *together* with its linear symplectomorphisms Γ. We would like to quantize not only the observables $\mathcal{A}$, but also the symmetries Γ. Next, we shall construct an equivariant quantization of $\mathbb{T}$.

2.2 Equivariant Weyl quantization of the torus

Let $\hbar = \frac{1}{p}$ and consider the additive character $\psi : \mathbb{F}_p \longrightarrow \mathbb{C}^*$, $\psi(t) = e^{\frac{2\pi it}{p}}$. We give here a slightly different presentation of the algebra $\mathcal{A}_\hbar$. Let $\mathcal{A}_\hbar$ be the free $\mathbb{C}$-algebra generated by the symbols $\{s(\xi) \mid \xi \in \Lambda^*\}$ and the relations $s(\xi + \eta) = \psi(\frac{1}{2}\omega(\xi, \eta))s(\xi)s(\eta)$. Here we consider ω as a map $\omega : \Lambda^* \times \Lambda^* \longrightarrow \mathbb{F}_p$. The lattice Λ^* serves as a standard basis for $\mathcal{A}_\hbar$:

$$s : \Lambda^* \longrightarrow \mathcal{A}_\hbar.$$

The group Γ acts on the lattice Λ^*. Therefore, it acts on $\mathcal{A}_\hbar$. It is easy to see that Γ acts on $\mathcal{A}_\hbar$ by homomorphisms of algebras. For an element $B \in \Gamma$, we denote by $f \longmapsto f^B$ the action of B on an element $f \in \mathcal{A}_\hbar$.

An equivariant quantization of the torus is a pair:

$$\begin{aligned} \pi_\hbar : \mathcal{A}_\hbar &\longrightarrow \mathrm{End}(\mathcal{H}_\hbar), \\ \rho_\hbar : \Gamma &\longrightarrow \mathrm{PGL}(\mathcal{H}_\hbar), \end{aligned}$$

where $\pi_\hbar$ is a representation of $\mathcal{A}_\hbar$ and $\rho_\hbar$ is a projective representation of Γ. These two should be compatible in the following manner:

$$\rho_\hbar(B)\pi_\hbar(f)\rho_\hbar(B)^{-1} = \pi_\hbar(f^B), \tag{2.2.1}$$

for every $B \in \Gamma$ and $f \in \mathcal{A}_\hbar$. Equation (2.2.1) is called the *Egorov identity*.

Let us suggest now a construction of an equivariant quantization of the torus.

Given a representation $\pi : \mathcal{A}_\hbar \longrightarrow \text{End}(\mathcal{H})$ and an element $B \in \Gamma$, we construct a new representation $\pi^B : \mathcal{A}_\hbar \longrightarrow \text{End}(\mathcal{H})$:

$$\pi^B(f) = \pi(f^B). \tag{2.2.2}$$

This gives an action of Γ on the set $\text{Irr}(\mathcal{A}_\hbar)$ of classes of irreducible representations. The set $\text{Irr}(\mathcal{A}_\hbar)$ has a very regular structure as a principal homogeneous space over $\mathbb{T}$. Moreover, every irreducible representation of $\mathcal{A}_\hbar$ is finite dimensional and of dimension p. The following theorem plays a central role in the construction.

Theorem 2.1 (Canonical invariant representation [GH1]) *Let $\hbar = \frac{1}{p}$, where p is a prime[1]. There exists a unique (up to isomorphism) irreducible representation $(\pi_\hbar, \mathcal{H}_\hbar)$ of $\mathcal{A}_\hbar$ for which its equivalence class is fixed by Γ.*

Let $(\pi_\hbar, \mathcal{H}_\hbar)$ be a representative of the fixed irreducible equivalence class. Then for every $B \in \Gamma$ we have:

$$\pi_\hbar^B \simeq \pi_\hbar. \tag{2.2.3}$$

This means that for every element $B \in \Gamma$ there exists an operator $\rho_\hbar(B)$ acting on $\mathcal{H}_\hbar$ which realizes the isomorphism (2.2.3). The collection $\{\rho_\hbar(B) : B \in \Gamma\}$ constitutes a projective representation:

$$\rho_\hbar : \Gamma \longrightarrow \text{PGL}(\mathcal{H}_\hbar). \tag{2.2.4}$$

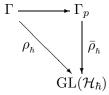Equations (2.2.2) and (2.2.3) also imply the Egorov identity (2.2.1).

The group $\Gamma \simeq \text{SL}_2(\mathbb{Z})$ is almost a free group and it is finitely presented. A brief analysis [GH1] shows that every projective representation of Γ can be lifted (linearized) into a true representation. More precisely, it can be linearized in 12 different ways, where 12 is the number of characters of Γ. In particular, the projective representation (2.2.4) can be linearized (*not* uniquely) into an honest representation. The next theorem (to be proved in Appendix B) proposes the existence of a canonical linearization. Let $\Gamma_p \simeq \text{SL}_2(\mathbb{F}_p)$ denotes the quotient group of Γ modulo p.

Theorem 2.2 (Canonical linearization) *Let $\hbar = \frac{1}{p}$, where $p \neq 2, 3$. There exists a unique linearization:*

$$\rho_\hbar : \Gamma \longrightarrow \text{GL}(\mathcal{H}_\hbar),$$

---

[1]The Theorem holds more generally, i.e., for all Planck constants of the form $\hbar = M/N$ where M, N are co-prime integers. However, in this paper we will not need to consider this generality

characterized by the property that it factors through the quotient group Γ_p:

$$
\begin{array}{ccc}
\Gamma & \longrightarrow & \Gamma_p \\
& \rho_\hbar \searrow & \downarrow \bar\rho_\hbar \\
& & \mathrm{GL}(\mathcal{H}_\hbar)
\end{array}
$$

From now on $\rho_\hbar$ means the linearization of Theorem 2.2.

Summary. Theorem 2.1 confirms the existence of a unique invariant representation of $\mathcal{A}_\hbar$, for every $\hbar = \frac{1}{p}$. This gives a canonical equivariant quantization $(\pi_\hbar, \rho_\hbar, \mathcal{H}_\hbar)$. Moreover, for $p \neq 2, 3$ by Theorem 2.2, the projective representation $\rho_\hbar$ can be linearized in a canonical way to give an honest representation of Γ which factors through $\Gamma_p{}^2$. Altogether this gives a pair:

$$
\begin{aligned}
\pi_\hbar : \mathcal{A}_\hbar &\longrightarrow \mathrm{End}(\mathcal{H}_\hbar), \\
\rho_\hbar : \Gamma_p &\longrightarrow \mathrm{GL}(\mathcal{H}_\hbar)
\end{aligned}
$$

satisfying the following compatibility condition (Egorov identity):

$$
\rho_\hbar(B)\pi_\hbar(f)\rho_\hbar(B)^{-1} = \pi_\hbar(f^B),
$$

for every $B \in \Gamma_p$, $f \in \mathcal{A}_\hbar$. The notation $\pi_\hbar(f^B)$ means that we take any pre-image $\bar{B} \in \Gamma$ of $B \in \Gamma_p$ and act by it on f. Note, that the operator $\pi_\hbar(f^{\bar{B}})$ does not depend on the choice of $\bar{B}$. In the following, we denote the Weil representation $\bar\rho_\hbar$ by $\rho_\hbar$ and consider Γ_p to be the default domain.

2.3   Quantum mechanical system

Let $(\pi_\hbar, \rho_\hbar, \mathcal{H}_\hbar)$ be the canonical equivariant quantization. Let A be our fixed hyperbolic element, considered as an element of Γ_p. The element A generates a quantum dynamical system as follows. Take a (pure) quantum state $\Psi \in S(\mathcal{H}_\hbar) = \{\Psi \in \mathcal{H}_\hbar : \|\Psi\| = 1\}$ and act on it with A:

$$
\Psi \longmapsto \Psi^A = \rho_\hbar(A)\Psi. \tag{2.3.1}
$$

3   Hecke Quantum Unique Ergodicity

The main silent question of this paper is whether the system (2.3.1) is quantum ergodic. Before discussing this question, one is obliged to define a notion of quantum ergodicity. As a first approximation, follow the classical definition, but replace each classical notion by its quantum

---

[2]This is the famous Weil representation of $\mathrm{SL}_2(\mathbb{F}_p)$.

counterpart. That is, for every $f \in \mathcal{A}_\hbar$ and almost every quantum state $\Psi \in S(\mathcal{H}_\hbar)$, the following holds:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} < \Psi | \pi_\hbar(f^{A^k}) \Psi > \overset{?}{=} \int_{\mathbb{T}} f|\omega|. \tag{3.0.2}$$

Unfortunately (3.0.2) is literally not true. The limit is never exactly equal to the integral for a fixed $\hbar$. Let us now give a true statement which is a slight modification of (3.0.2), called the *Hecke Quantum Unique Ergodicity*. First, rewrite (3.0.2) in an equivalent form. We have:

$$< \Psi | \pi_\hbar(f^{A^k}) \Psi > = < \Psi | \rho_\hbar(A^k) \pi_\hbar(f) \rho_\hbar(A^k)^{-1} \Psi >, \tag{3.0.3}$$

using the Egorov identity (2.2.1).

Now, note that the elements A^k run inside the finite group Γ_p. Denote by $\langle A \rangle \subseteq \Gamma_p$ the cyclic subgroup generated by A. It is easy to see, using (3.0.3), that:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} < \Psi | \pi_\hbar(f^{A^k}) \Psi > = \frac{1}{|\langle A \rangle|} \sum_{B \in \langle A \rangle} < \Psi | \rho_\hbar(B) \pi_\hbar(f) \rho_\hbar(B)^{-1} \Psi > .$$

Altogether (3.0.2) can be written in the form:

$$\mathbf{Av}_{\langle A \rangle}(< \Psi | \pi_\hbar(f) \Psi >) \overset{?}{=} \int_{\mathbb{T}} f|\omega|, \tag{3.0.4}$$

where $\mathbf{Av}_{\langle A \rangle}$ denotes the average of the *Wigner distribution* $< \Psi | \pi_\hbar(f) \Psi >$ with respect to the group $\langle A \rangle$.

3.1 Hecke theory

Denote by T_A the centralizer of A in $\Gamma_p \simeq \mathrm{SL}_2(\mathbb{F}_p)$. The finite group T_A consists of the rational points of an algebraic group $\mathbf{T}_A$. Moreover, in the case where the characteristic of the field does not divide $\mathrm{Tr}(A)^2 - 4$ the group $\mathbf{T}_A$ is an algebraic torus. We call T_A the *Hecke torus* (cf. [KR1]). One has, $\langle A \rangle \subseteq \mathrm{T}_A \subseteq \Gamma_p$. Now, in (3.0.4) take the average with respect to the group T_A instead of the group $\langle A \rangle$. The statement of the **Kurlberg-Rudnick rate conjecture** (cf. [KR1, R1, R2]) is given[3] in the following theorem:

Theorem 3.1 (Hecke Quantum Unique Ergodicity) *Let $\hbar = \frac{1}{p}$, where p is a sufficiently large prime[4]. For every $f \in \mathcal{A}_\hbar$ and $\Psi \in S(\mathcal{H}_\hbar)$, we have:*

$$\left| \mathbf{Av}_{\mathrm{T}_A}(< \Psi | \pi_\hbar(f) \Psi >) - \int_{\mathbb{T}} f|\omega| \right| \leq \frac{C_f}{\sqrt{p}}, \tag{3.1.1}$$

---

[3]Our conjecture is equivalent to the original statement [KR1], which treats only the case of common eigenstates of the Hecke torus.

[4]What one really needs here is that any non-trivial fixed element $\xi \in \Lambda^*$ will not be an eigenvector for the action of A on the quotient $\mathbb{F}_p$-vector space $\Lambda^*/p\Lambda^*$ for sufficiently large p. Hence, Theorem 3.1 holds true for every regular element in Γ that has no eigenvectors in the integral lattice Λ^*. The last property holds not only for hyperbolic elements! For example, Theorem 3.1 holds for the Weyl element $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$.

where C_f is an explicit constant depending only on f.

Section 4 is devoted to proving Theorem 3.1.

4 Proof of the Hecke Quantum Unique Ergodicity Conjecture

The proof is given in two stages. The first stage is a preparation stage and consists mainly of linear algebra considerations. We reduce statement (3.1.1) in several steps into an equivalent statement which will be better suited to our needs. In the second stage we introduce the main part of the proof, invoking tools from algebraic geometry in the framework of ℓ-adic sheaves and ℓ-adic cohomology (cf. [M, BBD]).

4.1 Preparation stage

Step 1. It is enough to prove Theorem 3.1 for the case when f is a non-trivial character $\xi \in \Lambda^*$. Because $\int_{\mathbb{T}} \xi |\omega| = 0$, statement (3.1.1) becomes :

$$\left| \mathbf{Av}_{\mathrm{T}_A}(< \Psi | \pi_\hbar(\xi) \Psi >) \right| \leq \frac{C_\xi}{\sqrt{p}}. \tag{4.1.1}$$

The statement for general $f \in \mathcal{A}_\hbar$ follows directly from the triangle inequality.

Step 2. It is enough to prove (4.1.1) in case $\Psi \in S(\mathcal{H}_\hbar)$ is a *Hecke* eigenstate. To be more precise, the *Hecke* torus T_A acts semisimply on $\mathcal{H}_\hbar$ via the representation $\rho_\hbar$, thus $\mathcal{H}_\hbar$ decomposes to a direct sum of character spaces:

$$\mathcal{H}_\hbar = \bigoplus_{\chi : \mathrm{T}_A \longrightarrow \mathbb{C}^*} \mathcal{H}_\chi. \tag{4.1.2}$$

The sum in (4.1.2) is over multiplicative characters of the torus T_A. For every $\Psi \in \mathcal{H}_\chi$ and $B \in \mathrm{T}_A$, we have:

$$\rho_\hbar(B)\Psi = \chi(B)\Psi.$$

Taking $\Psi \in \mathcal{H}_\chi$, statement (4.1.1) becomes:

$$|< \Psi | \pi_\hbar(\xi) \Psi >| \leq \frac{C_\xi}{\sqrt{p}}. \tag{4.1.3}$$

Here $C_\xi = 2 + o(1)$, where we use here the standard o notation.[5]

The averaged operator:

$$\mathbf{Av}_{\mathrm{T}_A}(\pi_\hbar(\xi)) = \frac{1}{|\mathrm{T}_A|} \sum_{B \in \mathrm{T}_A} \rho_\hbar(B)\pi_\hbar(\xi)\rho_\hbar(B)^{-1},$$

---

[5]In the case where T_A is non-split we have $C_\xi = 2$.

is essentially[6] diagonal in the Hecke base. Knowing this, statement (4.1.1) follows from (4.1.3) by invoking the triangle inequality.

Step 3. Let $P_\chi : \mathcal{H}_\hbar \longrightarrow \mathcal{H}_\hbar$ be the orthogonal projector on the eigenspace $\mathcal{H}_\chi$.

Remark 4.1 *For χ other then the quadratic character of T_A we have* $\dim \mathcal{H}_\chi = 1$.[7]

Using Remark 4.1 we can rewrite (4.1.3) in the form:

$$|\mathrm{Tr}(P_\chi \pi_\hbar(\xi))| \leq \frac{C_\xi}{\sqrt{p}}. \tag{4.1.4}$$

The projector P_χ can be defined in terms of the representation $\rho_\hbar$:

$$P_\chi = \frac{1}{|\mathrm{T}_A|} \sum_{B \in \mathrm{T}_A} \chi^{-1}(B) \rho_\hbar(B).$$

Now write (4.1.3) in the form:

$$\frac{1}{|\mathrm{T}_A|} \left| \sum_{B \in \mathrm{T}_A} \mathrm{Tr}(\rho_\hbar(B)\pi_\hbar(\xi))\chi^{-1}(B) \right| \leq \frac{C_\xi}{\sqrt{p}}. \tag{4.1.5}$$

On noting that $|\mathrm{T}_A| = p \pm 1$ and multiplying both sides of (4.1.5) by $|\mathrm{T}_A|$ we obtain that it is enough to prove the following statement:

Theorem 4.2 (Hecke Quantum Unique Ergodicity (Restated)) *Fix a non-trivial $\xi \in \Lambda^*$. Let $\hbar = \frac{1}{p}$, where p is a sufficiently large prime. For every character χ the following holds:*

$$\left| \sum_{B \in \mathrm{T}_A} \mathrm{Tr}(\rho_\hbar(B)\pi_\hbar(\xi))\chi(B) \right| \leq 2\sqrt{p}.$$

4.2   The trace function

We prove Theorem 4.2 using sheaf theoretic techniques. Before diving into geometric considerations, we investigate further the ingredients appearing in Theorem 4.2. Denote by F the function $F : \Gamma_p \times \Lambda^* \longrightarrow \mathbb{C}$ defined by:

$$F(B, \xi) = \mathrm{Tr}(\rho(B)\pi_\hbar(\xi)). \tag{4.2.1}$$

---

[6]This follows from Remark 4.1. If T_A does not split over $\mathbb{F}_p$ then $\mathbf{Av}_{\mathrm{T}_A}(\pi_\hbar(\xi))$ is diagonal in the Hecke basis. In case T_A splits then for the Legendre character σ we have that $\dim \mathcal{H}_\sigma = 2$. However, in the later case one can prove (4.1.1) for $v \in \mathcal{H}_\sigma$ by a computation of explicit eigenstates (cf. [KR2]).

[7]This fact, which is needed if we want to stick with the matrix coefficient formulation of the conjecture, can be proven by algebro-geometric techniques or alternatively by a direct computation (cf. [Ge]).

We denote by $V = \Lambda^*/p\Lambda^*$ the quotient vector space, i.e., $V \simeq \mathbb{F}_p^2$. The symplectic form ω specializes to give a symplectic form on V. The group Γ_p is the group of linear symplectomorphisms of V, i.e., $\Gamma_p = \mathrm{Sp}(V)$. Set $Y_0 = \Gamma_p \times \Lambda^*$ and $Y = \Gamma_p \times V$. One has (for a proof, see Section B.2) the quotient map:

$$Y_0 \longrightarrow Y.$$

Lemma 4.3 *The function* $F : Y_0 \longrightarrow \mathbb{C}$ *factors through the quotient* Y.

Denote the function $\overline{F}$ also by F and from now on Y will be considered as the default domain. The function $F : Y \longrightarrow \mathbb{C}$ is invariant under a certain group action of Γ_p. To be more precise, let $S \in \Gamma_p$. Then:

$$\mathrm{Tr}(\rho_\hbar(B)\pi_\hbar(\xi)) = \mathrm{Tr}(\rho_\hbar(S)\rho_\hbar(B)\rho_\hbar(S)^{-1}\rho_\hbar(S)\pi_\hbar(\xi)\rho_\hbar(S)^{-1}).$$

Applying the Egorov identity (2.2.1) and using the fact that $\rho_\hbar$ is a representation we get:

$$\mathrm{Tr}(\rho_\hbar(S)\rho_\hbar(B)\rho_\hbar(S)^{-1}\rho_\hbar(S)\pi_\hbar(\xi)\rho_\hbar(S)^{-1}) = \mathrm{Tr}(\pi_\hbar(S\xi)\rho_\hbar(SBS^{-1})).$$

Altogether we have:

$$F(B,\xi) = F(SBS^{-1}, S\xi). \qquad (4.2.2)$$

Putting (4.2.2) in a more diagrammatic form: there is an action of Γ_p on Y given by the following formula:

$$\Gamma_p \times Y \quad \xrightarrow{\ \alpha\ } \quad Y,$$
$$(S,(B,\xi)) \quad \longrightarrow \quad (SBS^{-1}, S\xi). \qquad (4.2.3)$$

Consider the following diagram:

$$Y \xleftarrow{\ pr\ } \Gamma_p \times Y \xrightarrow{\ \alpha\ } Y,$$

where pr is the projection on the Y variable. Formula (4.2.2) can be stated equivalently as:

$$\alpha^*(F) = pr^*(F),$$

where $\alpha^*(F)$ and $pr^*(F)$ are the pullbacks of the function F on Y via the maps α and pr respectively.

4.3 Geometrization (Sheafification)

Our next goal is to reduce Theorem 4.2 to a geometric statement, i.e., Lemma 4.6. The main tool which we invoke is called the "Geometrization" procedure. In this procedure one replaces sets by algebraic varieties and functions by sheaf theoretic objects (which are quite similar to vector bundles). The main statement of this section will be presented in the "Geometrization Theorem", i.e., Theorem 4.4.

4.3.1 Algebraic geometry

First, we have to devote some space recalling notions and notations from algebraic geometry over finite fields and the theory of ℓ-adic sheaves.

Varieties. In the sequel, we shall translate back and forth between algebraic varieties defined over the finite field $\mathbb{F}_p$, and their corresponding sets of rational points. In order to prevent confusion between the two, we use bold-face letters for denoting a variety $\mathbf{X}$, and normal letters for denoting its corresponding set of rational points X. An algebraic variety, which is defined over the finite field $\mathbb{F}_p$, is an algebraic variety equipped with an endomorphism called Frobenius:

$$\mathrm{Fr} : \mathbf{X} \longrightarrow \mathbf{X}.$$

This is also sometimes called *rational* structure. We denote by X the set of points fixed by the Frobenius, that is,:

$$\mathrm{X} = \mathbf{X}^{\mathrm{Fr}} = \{x \in \mathbf{X} : \mathrm{Fr}(x) = x\}.$$

Another common notation for this set is $\mathrm{X} = \mathbf{X}(\mathbb{F}_p)$.

In more detail, in this paper we decided not to use scheme theoretic language. Hence, all spaces considered are plain algebraic varieties defined over the algebraically closed field $\overline{\mathbb{F}}_p$, and points are morphisms $Spec(\overline{\mathbb{F}}_p) \to \mathbf{X}$. Given an algebraic variety $\mathbf{X}$, there exists the following Cartesian square:

$$
\begin{array}{ccc}
\mathbf{X}_0 & \xrightarrow{\ \mathrm{Fr}\ } & \mathbf{X} \\
\downarrow & & \downarrow \\
Spec(\overline{\mathbb{F}}_p) & \xrightarrow{\ \mathrm{Fr}\ } & Spec(\overline{\mathbb{F}}_p),
\end{array}
$$

where $\mathrm{Fr} : Spec(\overline{\mathbb{F}}_p) \to Spec(\overline{\mathbb{F}}_p)$, corresponds by duality to the Frobenius endomorphism of the field $\overline{\mathbb{F}}_p$. A variety $\mathbf{X}$ is said to be defined over the finite field $\mathbb{F}_p$, if it is equipped with an isomorphism $\nu : \mathbf{X} \to \mathbf{X}_0$. We denote the composition $\mathrm{Fr} \circ \nu : \mathbf{X} \to \mathbf{X}$, also by Fr.

Sheaves. Let $\mathrm{D}_{\mathrm{c}}^b(\mathbf{X})$ denote the "triangulated" category of constructible ℓ-adic sheaves on

X (cf. [M, BBD], in addition see [BL] for equivariant sheaves theory). We denote by $Perv(\mathbf{X})$ the Abelian category of perverse sheaves on **X**, that is the heart with respect to the autodual perverse t-structure in $\mathrm{D}^b_{\mathrm{c}}(\mathbf{X})$ (cf. [BBD]). We will use also the notion of N-perversity: an object $\mathcal{F}$ in $\mathrm{D}^b_{\mathrm{c}}(\mathbf{X})$ is called N-perverse if $\mathcal{F}[-N] \in Perv(\mathbf{X})$.

Finally, we define the notion of a Weil structure (or Frobenius structure). By a *Weil structure* on an object $\mathcal{F} \in \mathrm{D}^b_{\mathrm{c}}(\mathbf{X})$ we mean an isomorphism:

$$\theta : \mathrm{Fr}^* \mathcal{F} \simeq \mathcal{F}.$$

The pair $(\mathcal{F}, \theta)$ above is called a *Weil object*. By an abuse of notation we often denote θ also by Fr. We fix once an identification $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. Therefore, all sheaves are considered to be with coefficients over the complex numbers.

In the sequel we will use the following two standard sheaves. We denote by $\mathscr{L}_\psi$ the Artin-Schreier sheaf on the group $\mathbb{G}_a$ that corresponds to the additive character ψ on the group $\mathbb{F}_p = \mathbb{G}_a(\mathbb{F}_p)$, and by $\mathscr{L}_\sigma$ the Kummer sheaf on the multiplicative group $\mathbb{G}_m$ that corresponds to the Legendre quadratic character σ on the group $\mathbb{F}_p^* = \mathbb{G}_m(\mathbb{F}_p)$.

4.3.2 The Geometrization Theorem

Having said that, we can begin replacing all our sets with their corresponding algebraic varieties. The symplectic vector space (V, ω) is identified as the set of rational points of an algebraic variety **V**. The variety **V** is equipped with a morphism $\omega : \mathbf{V} \times \mathbf{V} \to \overline{\mathbb{F}}_p$ respecting the Frobenius structure on both sides. The group Γ_p is identified as the set of rational points of the algebraic group **Sp**. Finally, the set Y is identified as the set of rational points of the algebraic variety **Y**. More precisely, $\mathbf{Y} \simeq \mathbf{Sp} \times \mathbf{V}$. We denote by α the action of **Sp** on the variety **Y** (cf. (4.2.3)). We choose once an identification of the symplectic vector space with the standard symplectic plane:

$$(\mathbf{V}, \omega) \simeq (\mathbb{A}^2, \omega_{\mathrm{std}}), \tag{4.3.1}$$

where ω_{std} is the standard symplectic form defined by the condition $\omega_{\mathrm{std}}((1,0), (0,1)) = 1$. This induces an identification:

$$\mathbf{Sp} \simeq \mathbf{SL_2}. \tag{4.3.2}$$

Consider the standard torus $\mathbf{T} \subset \mathbf{SL_2}$, i.e., **T** consists of diagonal matrices of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix} \right)$. We use the notation $\mathbf{T}^\times$ for denoting the punctured torus $\mathbf{T} \smallsetminus \mathrm{I}$.

Our next goal is to replace functions by appropriate sheaf theoretic objects. The following theorem (for a proof, see Section B.3) proposes an appropriate sheaf theoretic object standing in place of the function $F : \mathrm{Y} \longrightarrow \mathbb{C}$ (4.2.1).

Theorem 4.4 (Geometrization Theorem) *There exists a Weil object $\mathcal{F} \in D_c^b(\mathbf{Y})$ satisfying the following properties:*

1. (Perversity) The object $\mathcal{F}$ is geometrically irreducible $\dim(\mathbf{Y})$-perverse of pure weight $w(\mathcal{F}) = 0$.[8]

2. (Function) The function F is associated to $\mathcal{F}$ via *sheaf-to-function correspondence*:
$$f^{\mathcal{F}} = F.$$

3. (Equivariance) For every element $S \in \mathbf{Sp}$ there exists an isomorphism:
$$\alpha_S^* \mathcal{F} \simeq \mathcal{F}.$$

4. (Formula) Restricting the sheaf $\mathcal{F}$ on the subvariety $\mathbf{T}^\times \times \mathbf{V}$. Using the identifications (4.3.1), (4.3.2). We have an explicit formula:
$$\mathcal{F}_{|\mathbf{T}^\times \times \mathbf{V}}\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right), \lambda, \mu\right) \simeq \mathscr{L}_{\sigma(a)} \otimes \mathscr{L}_{\psi(\frac{1}{2}\frac{a+1}{a-1}\lambda \cdot \mu)}. \tag{4.3.3}$$

Comments.

1. In Property 3, in fact a finer statement is true. In the case that $S \in \mathbf{Sp}(\mathbb{F}_q)$, $q = p^n$, one can show that the isomorphism $\alpha_S^* \mathcal{F} \simeq \mathcal{F}$ is an isomorphism of Weil sheaves on $\mathbf{Y}$, considered as an algebraic variety defined over $\mathbb{F}_q$.

2. In Property 4, one can produce an invariant formula, without using any identification. Moreover, this formula is valid on an open subvariety. We shall now explain this further. Let $j : \mathbf{U} \hookrightarrow \mathbf{Sp}$ denote the open subvariety consisting of elements $g \in \mathbf{Sp}$ such that $g - \mathrm{I}$ is invertible. Restricting the sheaf $\mathcal{F}$ to the open subvariety $\mathbf{U} \times \mathbf{V}$, we have the following isomorphism:
$$\mathcal{F}_{|\mathbf{U} \times \mathbf{V}} \simeq \mathscr{L}_{\psi(\frac{1}{4}\omega(\frac{g+\mathrm{I}}{g-\mathrm{I}}v,v))} \otimes \mathscr{L}_{\sigma(\mathrm{Tr}(g)-2)}.^{[9]} \tag{4.3.4}$$

It is a direct calculation to verify that the invariant formula (4.3.4) coincides with the coordinate dependent formula (4.3.3) when one restricts to the standard (punctured) torus $\mathbf{T}^\times$.

Explanation

---

[8] We thank the referee for bringing to our attention a deep observation regarding these striking properties.

[9] In particular, this implies that $\mathcal{F}$ coincides with the middle extension $j_{!*}(\mathscr{L}_{\psi(\frac{1}{4}\omega(\frac{g+\mathrm{I}}{g-\mathrm{I}}v,v))} \otimes \mathscr{L}_{\sigma(\mathrm{Tr}(g)-2)})$. This means that one obtains a complete description of the sheaf $\mathcal{F}$.

We give here an *intuitive* explanation of Theorem 4.4, part by part, as it was stated. An object $\mathcal{F} \in D_c^b(\mathbf{Y})$ can be considered as a vector bundle $\mathcal{F}$ over $\mathbf{Y}$:

$$\mathcal{F}$$
$$\downarrow$$
$$\mathbf{Y}.$$

Saying that $\mathcal{F}$ is a *Weil sheaf* means that it is equipped with a lifting of the Frobenius, that is,:

$$
\begin{array}{ccc}
\mathcal{F} & \xleftarrow{\ \mathrm{Fr}\ } & \mathcal{F} \\
\downarrow & & \downarrow \\
\mathbf{Y} & \xrightarrow{\ \mathrm{Fr}\ } & \mathbf{Y}
\end{array}
$$

Remark. We deliberately choose the lifting above in the opposite direction in order to make our intuitive explanation consistent with the formal definitions.

To be even more precise, think of $\mathcal{F}$ not as a single vector bundle, but as a complex $\mathcal{F} = \mathcal{F}^\bullet$ of vector bundles over $\mathbf{Y}$:

$$
\cdots \xrightarrow{\ d\ } \mathcal{F}^{-1} \xrightarrow{\ d\ } \mathcal{F}^0 \xrightarrow{\ d\ } \mathcal{F}^1 \xrightarrow{\ d\ } \cdots
$$

The complex $\mathcal{F}^\bullet$ is equipped with a lifting of Frobenius:

$$
\begin{array}{ccccccccc}
\cdots \xrightarrow{\ d\ } & \mathcal{F}^{-1} & \xrightarrow{\ d\ } & \mathcal{F}^0 & \xrightarrow{\ d\ } & \mathcal{F}^1 & \xrightarrow{\ d\ } & \cdots \\
& \downarrow{\scriptstyle\mathrm{Fr}} & & \downarrow{\scriptstyle\mathrm{Fr}} & & \downarrow{\scriptstyle\mathrm{Fr}} & & \\
\cdots \xrightarrow{\ d\ } & \mathcal{F}^{-1} & \xrightarrow{\ d\ } & \mathcal{F}^0 & \xrightarrow{\ d\ } & \mathcal{F}^1 & \xrightarrow{\ d\ } & \cdots
\end{array}
$$

Here the Frobenius commutes with the differentials. Next, we explain the meaning of Property 1. We will not try to explain here the notion of perversity, neither the notion of purity (cf. [BBD, D2]). Going into these matters will take us to far a part. It is enough for the purpose of this paper to explain the meaning of the sheaf $\mathcal{F}$ being of a mixed weight $w(\mathcal{F}) \leq 0$. This condition is implied by the condition of $\mathcal{F}$ being of pure weight $w(\mathcal{F}) = 0$. Let $y \in \mathrm{Y}$ be a point fixed by Frobenius. Denote by $\mathcal{F}_y$ the fiber of $\mathcal{F}$ at the point y. Thinking of $\mathcal{F}$ as a complex of vector bundles, it is clear what one means by taking the fiber at a point. The fiber $\mathcal{F}_y$ is just a complex of vector spaces. Because the point y is fixed by the Frobenius, it induces an endomorphism of $\mathcal{F}_y$:

$$
\begin{array}{ccccccccc}
\cdots \xrightarrow{\ d\ } & \mathcal{F}_y^{-1} & \xrightarrow{\ d\ } & \mathcal{F}_y^0 & \xrightarrow{\ d\ } & \mathcal{F}_y^1 & \xrightarrow{\ d\ } & \cdots \\
& \downarrow{\scriptstyle\mathrm{Fr}} & & \downarrow{\scriptstyle\mathrm{Fr}} & & \downarrow{\scriptstyle\mathrm{Fr}} & & \\
\cdots \xrightarrow{\ d\ } & \mathcal{F}_y^{-1} & \xrightarrow{\ d\ } & \mathcal{F}_y^0 & \xrightarrow{\ d\ } & \mathcal{F}_y^1 & \xrightarrow{\ d\ } & \cdots
\end{array}
\tag{4.3.5}
$$

The Frobenius acting as in (4.3.5) commutes with the differentials. Hence, it induces an action on cohomologies. For every $i \in \mathbb{Z}$ we have an endomorphism:

$$\mathrm{Fr} : \mathrm{H}^i(\mathcal{F}_y) \longrightarrow \mathrm{H}^i(\mathcal{F}_y). \tag{4.3.6}$$

Saying that an object $\mathcal{F}$ has mixed weight $w(\mathcal{F}) \leq w$ means that for every point $y \in \mathrm{Y}$ and for every $i \in \mathbb{Z}$ the absolute value of the eigenvalues of Fr acting on the i'th cohomology (4.3.6) satisfy:

$$\left| \mathrm{e.v}(\mathrm{Fr}\big|_{\mathrm{H}^i(\mathcal{F}_y)}) \right| \leq \sqrt{p}^{\,w+i}.$$

In our case $w = 0$ and therefore:

$$\left| \mathrm{e.v}(\mathrm{Fr}\big|_{\mathrm{H}^i(\mathcal{F}_y)}) \right| \leq \sqrt{p}^{\,i}. \tag{4.3.7}$$

Property 2 of Theorem 4.4 concerns a function $f^{\mathcal{F}} : \mathrm{Y} \longrightarrow \mathbb{C}$ associated to the sheaf $\mathcal{F}$. To define $f^{\mathcal{F}}$, we only have to describe its value at every point $y \in \mathrm{Y}$. For a given $y \in \mathrm{Y}$ the Frobenius acts on the cohomologies of the fiber $\mathcal{F}_y$ (cf. (4.3.6)). Now put:

$$f^{\mathcal{F}}(y) = \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{Tr}(\mathrm{Fr}\big|_{\mathrm{H}^i(\mathcal{F}_y)}).$$

In other words, the value $f^{\mathcal{F}}(y)$ is the alternating sum of traces of the operator Fr acting on the cohomologies of the fiber $\mathcal{F}_y$. This alternating sum is called the *Euler characteristic* of Frobenius and it is denoted by:

$$f^{\mathcal{F}}(y) = \chi_{\mathrm{Fr}}(\mathcal{F}_y).$$

Theorem 4.4 confirms that $f^{\mathcal{F}}$ is the trace function F defined earlier by formula (4.2.1). Associating the function $f^{\mathcal{F}}$ on the set $\mathbf{Y}^{\mathrm{Fr}}$ to the sheaf $\mathcal{F}$ on $\mathbf{Y}$ is a particular case of a general procedure called *Sheaf-to-Function Correspondence* [G]. As this procedure will be used later, next we spend some space explaining it in greater details (cf. [Ga]).

Grothendieck's Sheaf-to-Function Correspondence

Let $\mathbf{X}$ be an algebraic variety defined over $\mathbb{F}_q$. Let $\mathcal{L} \in \mathrm{D}^b_c(\mathbf{X})$ be a Weil sheaf. One can associate to $\mathcal{L}$ a function $f^{\mathcal{L}}$ on the set X by the following formula:

$$f^{\mathcal{L}}(x) = \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{Tr}(\mathrm{Fr}\big|_{\mathrm{H}^i(\mathcal{L}_x)}).$$

This procedure is called *Sheaf-To-Function correspondence*. Next, we describe some important functorial properties of this procedure.

Let $\mathbf{X}_1$, $\mathbf{X}_2$ be algebraic varieties defined over $\mathbb{F}_q$. Let $\mathrm{X}_1 = \mathbf{X}_1^{\mathrm{Fr}}$ and $\mathrm{X}_2 = \mathbf{X}_2^{\mathrm{Fr}}$ be the corresponding sets of rational points. Let $\pi : \mathbf{X}_1 \longrightarrow \mathbf{X}_2$ be a morphism of algebraic varieties. Denote also by $\pi : \mathrm{X}_1 \longrightarrow \mathrm{X}_2$ the induced map on the level of sets.

First statement. Suppose we have a Weil sheaf $\mathcal{L} \in D_c^b(\mathbf{X}_2)$. The following holds:

$$f^{\pi^*(\mathcal{L})} = \pi^*(f^{\mathcal{L}}), \tag{4.3.8}$$

where on the function level π^* is just the pull back of functions. On the sheaf theoretic level π^* is the pull-back functor of sheaves (think of pulling back a vector bundle). Equation (4.3.8) states that the *Sheaf-to-Function Correspondence* commutes with the operation of pull back.

Second statement. Suppose we have a Weil sheaf $\mathcal{L} \in D_c^b(\mathbf{X}_1)$. The following holds:

$$f^{\pi_!(\mathcal{L})} = \pi_!(f^{\mathcal{L}}), \tag{4.3.9}$$

where on the function level, $\pi_!$ means to sum up the values of the function along the fibers of the map π. On the sheaf theoretic level, $\pi_!$ is a compact integration of sheaves (here we have no analogue under the vector bundle interpretation). Equation (4.3.9) states that the *Sheaf-to-Function Correspondence* commutes with integration.

Third statement. Suppose we have two Weil sheaves $\mathcal{L}_1, \mathcal{L}_2 \in D_c^b(\mathbf{X}_1)$. The following holds:

$$f^{\mathcal{L}_1 \otimes \mathcal{L}_2} = f^{\mathcal{L}_1} \cdot f^{\mathcal{L}_2}. \tag{4.3.10}$$

In other words, *Sheaf-to-Function Correspondence* takes tensor product of sheaves to multiplication of the corresponding functions.

4.4 Geometric statement

Fix an element $\xi \in \Lambda^*$ and a sufficiently large prime p so that $\xi \pmod{p}$ is not a T_A-eigenvector[10]. We denote by i_ξ the inclusion map $i_\xi : T_A \times \xi \longrightarrow Y$. Returning to Theorem 4.2 and putting its content in a functorial notation, we write the following inequality:

$$\left| pr_!(i_\xi^*(F) \cdot \chi) \right| \leq 2\sqrt{p}.$$

In other words, taking the function $F : Y \longrightarrow \mathbb{C}$ and:

- Restrict F to $T_A \times \xi$ and get $i_\xi^*(F)$.

- Multiply $i_\xi^* F$ by the character χ to get $i_\xi^*(F) \cdot \chi$.

- Integrate $i_\xi^*(F) \cdot \chi$ to the point, this means to sum up all its values, and get a scalar $a_\chi = pr_!(i_\xi^*(F) \cdot \chi)$. Here pr stands for the projection $pr : T_A \times \xi \longrightarrow pt$.

---

[10]Note that this can be done for every $\xi \in \Lambda^*$ due the fact that $A \in \mathrm{SL}_2(\mathbb{Z})$ is an hyperbolic element and does not have eigenvectors in Λ^*.

Then Theorem 4.2 asserts that the scalar a_χ is of an absolute value less than $2\sqrt{p}$.

Repeat the same steps in the geometric setting. We denote again by i_ξ the closed imbedding $i_\xi : \mathbf{T}_A \times \xi \longrightarrow \mathbf{Y}$. Take the sheaf $\mathcal{F}$ on $\mathbf{Y}$ and apply the following sequence of operations:

- Pull-back $\mathcal{F}$ to the closed subvariety $\mathbf{T}_A \times \xi$ and get the sheaf $i_\xi^*(\mathcal{F})$.

- Take the tensor product of $i_\xi^*(\mathcal{F})$ with the Kummer sheaf $\mathscr{L}_\chi$ and get $i_\xi^*(\mathcal{F}) \otimes \mathscr{L}_\chi$.

- Integrate $i_\xi^*(\mathcal{F}) \otimes \mathscr{L}_\chi$ to the point and get the sheaf $pr_!(i_\xi^*(\mathcal{F}) \otimes \mathscr{L}_\chi)$ on the point.

The operation of *Sheaf-to-Function Correspondence* commutes both with pullback (4.3.8), with integration (4.3.9) and takes the tensor product of sheaves to the multiplication of functions (4.3.10). This means that it intertwines the operations carried out on the level of sheaves with those carried out on the level of functions. The following diagram describes pictorially what has been said so far:

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\;\chi_{\mathrm{Fr}}\;} & F \\
i_\xi \uparrow & & i_\xi \uparrow \\
i_\xi^*(\mathcal{F}) \otimes \mathscr{L}_\chi & \xrightarrow{\;\chi_{\mathrm{Fr}}\;} & i_\xi^*(F) \cdot \chi \\
pr \downarrow & & pr \downarrow \\
pr_!(i_\xi^*(\mathcal{F}) \otimes \mathscr{L}_\chi) & \xrightarrow{\;\chi_{\mathrm{Fr}}\;} & pr_!(i_\xi^*(F) \cdot \chi).
\end{array}
$$

Recall the weight property $w(\mathcal{F}) \le 0$. Now, the effect of functors i_ξ^*, $pr_!$ and tensor product $\otimes$ on the property of weight should be examined.

The functor i_ξ^* does not increase weight. Observing the definition of weight this claim is immediate. Therefore, we get:
$$w(i_\xi^*(\mathcal{F})) \le 0.$$
Assume we have two sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ of mixed weights $w(\mathcal{L}_1) \le w_1$ and $w(\mathcal{L}_2) \le w_2$. Then the weight of the tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2$ is of mixed weight $w(\mathcal{L}_1 \otimes \mathcal{L}_2) \le w_1 + w_2$. This is again immediate from the definition of weight.

Knowing that the Kummer sheaf has pure weight $w(\mathscr{L}_\chi) = 0$, we deduce:
$$w(i_\xi^*(\mathcal{F}) \otimes \mathscr{L}_\chi) \le 0.$$

Finally, one has to understand the effect of the functor $pr_!$. The following theorem, proposed by Deligne [D2], is a very deep and important result in the theory of weights. Briefly speaking, the theorem states that compact integration of sheaves does not increase weight. Here is the precise statement:

Theorem 4.5 (Deligne, Weil II [D2]) *Let $\pi : \mathbf{X}_1 \longrightarrow \mathbf{X}_2$ be a morphism of algebraic varieties. Let $\mathcal{L} \in \mathrm{D}_c^b(\mathbf{X}_1)$ be a sheaf of mixed weight $w(\mathcal{L}) \le w$ then the sheaf $\pi_!(\mathcal{L})$ is of mixed weight $w(\pi_!(\mathcal{L})) \le w$.*

Using Theorem 4.5 we get:
$$w(pr_!(i_\xi^*(\mathcal{F}) \otimes \mathcal{L}_\chi)) \le 0.$$

Consider the sheaf $\mathcal{G} = pr_!(i_\xi^*(\mathcal{F}) \otimes \mathcal{L}_\chi)$. It is an object in $D_c^b(pt)$. This means it is merely a complex of vector spaces, $\mathcal{G} = \mathcal{G}^\bullet$, together with an action of Frobenius:

$$
\begin{array}{ccccccccc}
\cdots & \xrightarrow{d} & \mathcal{G}^{-1} & \xrightarrow{d} & \mathcal{G}^0 & \xrightarrow{d} & \mathcal{G}^1 & \xrightarrow{d} & \cdots \\
& & \downarrow{\scriptstyle \mathrm{Fr}} & & \downarrow{\scriptstyle \mathrm{Fr}} & & \downarrow{\scriptstyle \mathrm{Fr}} & & \\
\cdots & \xrightarrow{d} & \mathcal{G}^{-1} & \xrightarrow{d} & \mathcal{G}^0 & \xrightarrow{d} & \mathcal{G}^1 & \xrightarrow{d} & \cdots
\end{array}
$$

The complex $\mathcal{G}^\bullet$ is associated by *Sheaf-To-Function correspondence* to the scalar a_χ:

$$a_\chi = \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{Tr}(\mathrm{Fr}\big|_{\mathrm{H}^i(\mathcal{G})}). \tag{4.4.1}$$

Finally, we can give the geometric statement about $\mathcal{G}$, which will imply Theorem 4.2.

Lemma 4.6 (Vanishing Lemma) *Let $\mathcal{G} = pr_!(i_\xi^*(\mathcal{F}) \otimes \mathcal{L}_\chi)$, where ξ is not a T_A-eigenvector. All cohomologies $\mathrm{H}^i(\mathcal{G})$ vanish except for $i = 1$. Moreover, $\mathrm{H}^1(\mathcal{G})$ is a two-dimensional vector space.*

Theorem 4.2 now follows easily. By Lemma 4.6 only the first cohomology $\mathrm{H}^1(\mathcal{G})$ does not vanish and it is two-dimensional. Having $w(\mathcal{G}) \le 0$ implies (using (4.3.7)) that the eigenvalues of Frobenius acting on $\mathrm{H}^1(\mathcal{G})$ are of absolute value $\le \sqrt{p}$. Hence, using formula (4.4.1) we get:

$$|a_\chi| \le 2\sqrt{p}.$$

The remainder of this section is devoted to the proof of Lemma 4.6.

4.5 Proof of the Vanishing Lemma

The proof will be given in several steps.

Step 1. We use the identifications (4.3.1), and (4.3.2). Note that all tori in $\mathbf{SL_2}$ are conjugated. Therefore, there exists an element $S \in \mathbf{SL_2}$ conjugating the *Hecke* torus $\mathbf{T}_A \subset \mathbf{SL_2}$ with the standard torus $\mathbf{T}$:
$$S\mathbf{T}_A S^{-1} = \mathbf{T}.$$
The situation is displayed in the following diagram:

$$
\begin{array}{ccc}
\mathbf{SL_2} \times \mathbb{A}^2 & \xrightarrow{\alpha_S} & \mathbf{SL_2} \times \mathbb{A}^2 \\
{\scriptstyle i_\xi} \uparrow & & {\scriptstyle i_\eta} \uparrow \\
\mathbf{T}_A \times \xi & \xrightarrow{\alpha_S} & \mathbf{T} \times \eta \\
{\scriptstyle pr} \downarrow & & {\scriptstyle pr} \downarrow \\
pt & =\!=\!= & pt
\end{array}
$$

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where $\eta = S \cdot \xi$ and α_S is the restriction of the action α to the element S.

Step 2. Using the equivariance property of the sheaf $\mathcal{F}$ (see Theorem 4.4, Property 3) we will show that it is *sufficient* to prove the Vanishing Lemma for the sheaf $\mathcal{G}_{st} = pr_!(i_\eta^* \mathcal{F} \otimes \alpha_{S!} \mathscr{L}_\chi)$.

Indeed, we have:
$$\mathcal{G} = pr_!(i_\xi^* \mathcal{F} \otimes \mathscr{L}_\chi) \simeq pr_! \alpha_{S!}(i_\xi^* \mathcal{F} \otimes \mathscr{L}_\chi). \tag{4.5.1}$$

The morphism α_S is an isomorphism. Therefore, $\alpha_{S!}$ commutes with taking $\otimes$, hence we obtain:
$$pr_! \alpha_{S!}(i_\xi^*(\mathcal{F}) \otimes \mathscr{L}_\chi) \simeq pr_!(\alpha_{S!}(i_\xi^* \mathcal{F}) \otimes \alpha_{S!}(\mathscr{L}_\chi)). \tag{4.5.2}$$

Applying base change we obtain:
$$\alpha_{S!} i_\xi^* \mathcal{F} \simeq i_\eta^* \alpha_{S!} \mathcal{F}. \tag{4.5.3}$$

Now using the equivariance property of the sheaf $\mathcal{F}$ we have the isomorphism:
$$\alpha_{S!} \mathcal{F} \simeq \mathcal{F}. \tag{4.5.4}$$

Combining (4.5.1), (4.5.2), (4.5.3) and (4.5.4) we get:
$$pr_!(i_\xi^* \mathcal{F} \otimes \mathscr{L}_\chi) \simeq pr_!(i_\eta^* \mathcal{F} \otimes \alpha_{S!} \mathscr{L}_\chi). \tag{4.5.5}$$

Therefore, we see from (4.5.5) that it is sufficient to prove vanishing of cohomologies for:
$$\mathcal{G}_{st} = pr_!(i_\eta^* \mathcal{F} \otimes \alpha_{S!} \mathscr{L}_\chi). \tag{4.5.6}$$

However, this is a situation over the standard torus and we can compute explicitly all the sheaves involved!

Step 3. The Vanishing Lemma holds for the sheaf $\mathcal{G}_{st}$.

What remains is to compute (4.5.6). We write $\eta = (\lambda, \mu)$. By Theorem 4.4 Property 4, we have $i_\eta^* \mathcal{F} \simeq \mathscr{L}_{\psi(\frac{1}{2} \frac{a+1}{a-1} \lambda \cdot \mu)} \otimes \mathscr{L}_{\sigma(a)}$, where a is the coordinate of the standard torus $\mathbf{T}$ and $\lambda \cdot \mu \neq 0$[11]. The sheaf $\alpha_{S!} \mathscr{L}_\chi$ is a character sheaf on the torus $\mathbf{T}$. Hence we get that (4.5.6) is a kind of a Kloosterman-sum sheaf. A direct computation (Appendix B section B.4) proves the Vanishing Lemma for this sheaf. This completes the proof of the Hecke quantum unique ergodicity conjecture. ∎

Comment. We mention that, in this paper we obtained also an alternative proof of the Vanishing Lemma, i.e., using the invariant formula for the sheaf $\mathcal{F}$ (see Formula (4.3.4)). Indeed, noting that the invariant formula of the sheaf $\mathcal{F}$ is valid on an open subvariety $\mathbf{U} \subset \mathbf{Sp} \times \mathbf{V}$, which contains $\mathbf{T}_A^\times \times \mathbf{V}$, we proceed and prove the statement directly, without the need to use the equivariance property, and with essentially the same cohomological computations.

---

[11] Recall that η is not a T-eigenvector.

Appendix

A   Metaplectique

In the first part of this Appendix we give new construction of the *Weil* (*metaplectic*) *representation* $(\rho, \mathrm{Sp}(\mathrm{V}), \mathcal{H}_\mathrm{V})$, attached to a two-dimensional symplectic vector space (V, ω) over $\mathbb{F}_q$, which appears in the body of the paper. The difference is that here the construction is slightly more general. Even more importantly, it is obtained in completely natural geometric terms. The focal step in our approach is the introduction of a *canonical Hilbert space* on which the Weil representation is naturally manifested. The motivation to look for this space was initiated by a question of David Kazhdan [Ka]. The key idea behind this construction was suggested to us by Joseph Bernstein [B]. The upshot is to replace the notion of a Lagrangian subspace by a more refined notion of an *oriented Lagrangian subspace*[12].

In the second part of this Appendix we apply a *geometrization* procedure to the construction given in the first part, meaning that all sets are replaced by algebraic varieties and all functions are replaced by ℓ-adic sheaves. This part is based on a letter of Deligne to Kazhdan from 1982 [D1]. We extract from that work only the part that is most relevant to this paper. Although all basic ideas appear already in the letter, we tried to give here a slightly more general and detailed account of the construction. As far as we know, the contents of this mathematical work has never been published. This might be a good enough reason for writing this part.

The following is a description of the Appendix. In section A.1 we introduce the notion of oriented Lagrangian subspace and the construction of the canonical Hilbert space. In section A.2 we obtain a natural realization of the Weil representation. In section A.3 we give the standard Schrödinger realization (cf. [Ge, H, W2]). We also include several formulas for the kernels of basic operators. These formulas will be used in section A.4 where the geometrization procedure is described. In section A.5 we give proofs of all lemmas and propositions which appear in previous sections.

For the remainder of the Appendix we fix the following notations. Let $\mathbb{F}_q$ denotes the finite field of characteristic $p \neq 2$ and q elements. Fix $\psi : \mathbb{F}_q \longrightarrow \mathbb{C}^*$ a non-trivial additive character. Denote by $\sigma : \mathbb{F}_q^* \longrightarrow \mathbb{C}^*$ the *Legendre* multiplicative quadratic character.

---

[12]We thank A. Polishchuk for pointing out to us that this is an $\mathbb{F}_q$-analogue of well known considerations, due to Lion and Vergne [LV], with usual oriented Lagrangians giving explicitly the metaplectic covering of $\mathrm{Sp}(2n, \mathbb{R})$.

A.1 Canonical Hilbert space

A.1.1 Oriented Lagrangian subspace

Let (V, ω) be a 2-dimensional symplectic vector space over $\mathbb{F}_q$.

Definition A.1 *An oriented Lagrangian subspace is a pair* (L, ϱ_L), *where* L *is a Lagrangian subspace of* V *and* $\varrho_L : L \smallsetminus \{0\} \longrightarrow \{\pm 1\}$ *is a function which satisfies the following equivariant property:*

$$\varrho_L(t \cdot l) = \sigma(t)\varrho_L(l),$$

where $t \in \mathbb{F}_q^*$ *and* σ *the Legendre character of* $\mathbb{F}_q^*$.

We denote by Lag° the space of oriented Lagrangians subspaces. There is a forgetful map $\mathrm{Lag}^\circ \longrightarrow \mathrm{Lag}$, where Lag is the space of Lagrangian subspaces, $\mathrm{Lag} \simeq \mathbb{P}^1(\mathbb{F}_q)$. In the sequel we use the notation L° to specify that L is equipped with an orientation.

A.1.2 The Heisenberg group

Let H be the Heisenberg group. As a set we have $H = V \times \mathbb{F}_q$. The multiplication is defined by the following formula:

$$(v, \lambda) \cdot (v', \lambda') = (v + v', \lambda + \lambda' + \tfrac{1}{2}\omega(v, v')).$$

We have a projection $\pi : H \longrightarrow V$. We fix a section of this projection:

$$s : V \dashrightarrow H, \quad s(v) = (v, 0). \tag{A.1.1}$$

A.1.3 Models of irreducible representation

Given $L^\circ = (L, \varrho_L) \in \mathrm{Lag}^\circ$, we construct the Hilbert space $\mathcal{H}_{L^\circ} = \mathrm{Ind}_{\tilde{L}}^{H} \mathbb{C}_{\tilde{\psi}}$, where $\tilde{L} = \pi^{-1}(L)$ and $\tilde{\psi}$ is the extension of the additive character ψ to $\tilde{L}$ using the section s, i.e., $\tilde{\psi} : \tilde{L} = L \times \mathbb{F}_q \longrightarrow \mathbb{C}^*$ is given by the formula:

$$\tilde{\psi}(l, \lambda) = \psi(\lambda).$$

More concretely: $\mathcal{H}_{L^\circ} = \{f : H \longrightarrow \mathbb{C} \mid f(\lambda l h) = \psi(\lambda)f(h)\}$. The group H acts on $\mathcal{H}_{L^\circ}$ by multiplication from the right. It is well known (and easy to prove) that the representations $\mathcal{H}_{L^\circ}$ of H are irreducible and for different L°'s they are all isomorphic. These are different models of the same irreducible representation. This is stated in the following theorem:

Theorem A.2 (Stone-von Neumann) *For an oriented Lagrangian subspace* L°, *the representation* $\mathcal{H}_{L^\circ}$ *of* H *is irreducible. Moreover, for any two oriented Lagrangians* $L_1^\circ, L_2^\circ \in \mathrm{Lag}^\circ$ *one has* $\mathcal{H}_{L_1^\circ} \simeq \mathcal{H}_{L_2^\circ}$ *as representations of* H. $\blacksquare$

A.1.4 Canonical intertwining operators

Let $L_1^\circ, L_2^\circ \in \mathrm{Lag}^\circ$ be two oriented Lagrangians. Let $\mathcal{H}_{L_1^\circ}, \mathcal{H}_{L_2^\circ}$ be the corresponding representations of H. We denote by $\mathrm{Int}_{L_2^\circ, L_1^\circ} = \mathrm{Hom}_H(\mathcal{H}_{L_1^\circ}, \mathcal{H}_{L_2^\circ})$ the space of intertwining operators between the two representations. Because all representations are irreducible and isomorphic to each other we have $\dim(\mathrm{Int}_{L_2^\circ, L_1^\circ}) = 1$. Next, we construct a canonical element in $\mathrm{Int}_{L_2^\circ, L_1^\circ}$.

Let $L_1^\circ = (L_1, \varrho_{L_1})$, $L_2^\circ = (L_2, \varrho_{L_2})$ be two oriented Lagrangian subspaces. Assume they are in general position, i.e., $L_1 \neq L_2$. We define the following specific element $\mathsf{F}_{L_2^\circ, L_1^\circ} \in \mathrm{Int}_{L_2^\circ, L_1^\circ}$, $\mathsf{F}_{L_2^\circ, L_1^\circ} : \mathcal{H}_{L_1^\circ} \longrightarrow \mathcal{H}_{L_2^\circ}$. It is defined by the following formula:

$$\mathsf{F}_{L_2^\circ, L_1^\circ} = \mathrm{a}_{L_2^\circ, L_1^\circ} \cdot \tilde{\mathsf{F}}_{L_2^\circ, L_1^\circ}, \tag{A.1.2}$$

where $\tilde{\mathsf{F}}_{L_2^\circ, L_1^\circ} : \mathcal{H}_{L_1^\circ} \longrightarrow \mathcal{H}_{L_2^\circ}$ denotes the standard averaging operator and $\mathrm{a}_{L_2^\circ, L_1^\circ}$ denotes the normalization factor. The formulas are:

$$\tilde{\mathsf{F}}_{L_2^\circ, L_1^\circ}(f)(h) = \sum_{l_2 \in L_2} f(l_2 h),$$

where $f \in \mathcal{H}_{L_1^\circ}$.

$$\mathrm{a}_{L_2^\circ, L_1^\circ} = \frac{1}{q} \sum_{l_1 \in L_1} \psi(\tfrac{1}{2}\omega(l_1, \xi_{L_2}))\varrho_{L_1}(l_1)\varrho_{L_2}(\xi_{L_2}),$$

where ξ_{L_2} is a fixed non-zero vector in L_2. Note that $\mathrm{a}_{L_2^\circ, L_1^\circ}$ does not depend on ξ_{L_2}.

Now we extend the definition of $\mathsf{F}_{L_2^\circ, L_1^\circ}$ to the case where $L_1 = L_2$. Define:

$$\mathsf{F}_{L_2^\circ, L_1^\circ} = \begin{cases} \mathrm{I}, & \varrho_{L_1} = \varrho_{L_2} \\ -\mathrm{I}, & \varrho_{L_1} = -\varrho_{L_2} \end{cases}$$

The *main claim* is that the collection $\{\mathsf{F}_{L_2^\circ, L_1^\circ}\}_{L_1^\circ, L_2^\circ \in \mathrm{Lag}^\circ}$ is associative. This is formulated in the following theorem:

Theorem A.3 (Associativity) *Let* $L_1^\circ, L_2^\circ, L_3^\circ \in \mathrm{Lag}^\circ$ *be a triple of oriented Lagrangian subspaces. The following associativity condition holds:*

$$\mathsf{F}_{L_3^\circ, L_2^\circ} \circ \mathsf{F}_{L_2^\circ, L_1^\circ} = \mathsf{F}_{L_3^\circ, L_1^\circ}.$$

For a proof, see Section A.5.

A.1.5 Canonical Hilbert space

Define the *canonical Hilbert space* $\mathcal{H}_V \subset \bigoplus_{L^\circ \in \mathrm{Lag}^\circ} \mathcal{H}_{L^\circ}$ as the subspace of compatible systems of vectors, that is,:

$$\mathcal{H}_V = \{(f_{L^\circ})_{L^\circ \in \mathrm{Lag}^\circ}; \quad \mathsf{F}_{L_2^\circ, L_1^\circ}(f_{L_1^\circ}) = f_{L_2^\circ}\}.$$

A.2 The Weil representation

In this section we construct the Weil representation using the Hilbert space $\mathcal{H}_V$. We denote by $\mathrm{Sp} = \mathrm{Sp}(V, \omega)$ the group of linear symplectomorphisms of V. Before giving any formulas, note that the space $\mathcal{H}_V$ was constructed out of the symplectic space (V, ω) in a complete canonical way. This immediately implies that all the symmetries of (V, ω) automatically acts on $\mathcal{H}_V$. In particular, we obtain a *linear* representation of the group Sp in the space $\mathcal{H}_V$. This is the famous Weil representation of Sp and we denote it by $\rho : \mathrm{Sp} \longrightarrow \mathrm{GL}(\mathcal{H}_V)$. It is given by the following formula:

$$\rho(g)[(f_{L^\circ})] = (f^g_{L^\circ}). \tag{A.2.1}$$

Let us elaborate on this formula. The group Sp acts on the space Lag°. Any element $g \in \mathrm{Sp}$ induces an automorphism $g : \mathrm{Lag}^\circ \longrightarrow \mathrm{Lag}^\circ$ defined by:

$$(L, \varrho_L) \longmapsto (gL, \varrho_L^g),$$

where $\varrho_L^g(l) = \varrho_L(g^{-1}l)$. Moreover, g induces an isomorphism of vector spaces $g : \mathcal{H}_{L^\circ} \longrightarrow \mathcal{H}_{gL^\circ}$ defined by the following formula:

$$f_{L^\circ} \longmapsto f^g_{L^\circ}, \quad f^g_{L^\circ}(h) = f_{L^\circ}(g^{-1}h), \tag{A.2.2}$$

where the action of $g \in \mathrm{Sp}$ on $h = (v, \lambda) \in \mathrm{H}$ is given by $g(v, \lambda) = (gv, \lambda)$. It is easy to verify that the action (A.2.2) of Sp commutes with the canonical intertwining operators, that is, for any two $L_1^\circ, L_2^\circ \in \mathrm{Lag}^\circ$ and any element $g \in \mathrm{Sp}$ the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{H}_{L_1^\circ} & \xrightarrow{\ F_{L_2^\circ, L_1^\circ}\ } & \mathcal{H}_{L_2^\circ} \\
{\scriptstyle g}\Big\downarrow & & {\scriptstyle g}\Big\downarrow \\
\mathcal{H}_{gL_1^\circ} & \xrightarrow{\ F_{gL_2^\circ, gL_1^\circ}\ } & \mathcal{H}_{gL_2^\circ}.
\end{array}
$$

From this we deduce that formula (A.2.1) indeed gives the action of Sp on $\mathcal{H}_V$.

A.3 Realization and formulas

In this section we give the standard Schrödinger realization of the Weil representation. Several formulas for the kernels of basic operators are also included.

A.3.1 Schrödinger realization

Fix $V = V_1 \oplus V_2$ to be a Lagrangian decomposition of V. Fix ϱ_{V_2} to be an orientation on V_2. Denote by $V_2^\circ = (V_2, \varrho_{V_2})$ the oriented space. Using the system of canonical intertwining

operators we identify $\mathcal{H}_V$ with a specific representative $\mathcal{H}_{V_2^\circ}$. Using the section $s : V \dashrightarrow H$ (cf. A.1.1) we further make the identification $s : \mathcal{H}_{V_2^\circ} \simeq \mathcal{S}(V_1)$, where $\mathcal{S}(V_1)$ is the space of complex valued functions on V_1. We denote $\mathcal{H} = \mathcal{S}(V_1)$. In this realization the Weil representation, $\rho : \mathrm{Sp} \longrightarrow \mathrm{GL}(\mathcal{H})$, is given by the following formula:

$$\rho(g)(f) = \mathsf{F}_{V_2^\circ, gV_2^\circ}(f^g),$$

where $f \in \mathcal{H} \simeq \mathcal{H}_{V_2^\circ}$ and $g \in \mathrm{Sp}$.

A.3.2 Formulas for the Weil representation

First we introduce a basis $e \in V_1$ and the dual basis $e^* \in V_2$ normalized so that $\omega(e, e^*) = 1$. In terms of this basis we have the following identifications: $V \simeq \mathbb{F}_q^2$, $V_1, V_2 \simeq \mathbb{F}_q$, $\mathrm{Sp} \simeq \mathrm{SL}_2(\mathbb{F}_q)$ and $H \simeq \mathbb{F}_q^2 \times \mathbb{F}_q$ (as sets). We also have $\mathcal{H} \simeq \mathcal{S}(\mathbb{F}_q)$.

For every element $g \in \mathrm{Sp}$ the operator $\rho(g) : \mathcal{H} \longrightarrow \mathcal{H}$ is represented by a kernel $K_g : \mathbb{F}_q^2 \longrightarrow \mathbb{C}$. The multiplication of operators becomes convolution of kernels. The collection $\{K_g\}_{g \in \mathrm{Sp}}$ gives a single function of "kernels" which we denote by $K_\rho : \mathrm{Sp} \times \mathbb{F}_q^2 \longrightarrow \mathbb{C}$. For every element $g \in \mathrm{Sp}$ the kernel $K_\rho(g)$ is of the form:

$$K_\rho(g, x, y) = a_g \cdot \psi(R_g(x, y)),$$

where a_g is a certain normalizing coefficient and $R_g : \mathbb{F}_q^2 \longrightarrow \mathbb{F}_q$ is a quadratic function supported on some linear subspace of $\mathbb{F}_q^2$. Next, we give an explicit description of the kernels $K_\rho(g)$.

Consider the (opposite) Bruhat decomposition $\mathrm{Sp} = \mathrm{B}w\mathrm{B} \cup \mathrm{B}$ where:

$$\mathrm{B} = \begin{pmatrix} * & \\ * & * \end{pmatrix},$$

and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the standard Weyl element.

If $g \in \mathrm{B}w\mathrm{B}$ then:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $b \neq 0$. In this case we have:

$$a_g = \frac{1}{q} \sum_{t \in \mathbb{F}_q} \psi(\frac{b}{2}t)\sigma(t),$$

$$R_g(x, y) = \frac{-b^{-1}d}{2}x^2 + \frac{b^{-1} - c + ab^{-1}d}{2}xy - \frac{ab^{-1}}{2}y^2.$$

Altogether we have:

$$K_\rho(g, x, y) = a_g \cdot \psi(\frac{-b^{-1}d}{2}x^2 + \frac{b^{-1} - c + ab^{-1}d}{2}xy - \frac{ab^{-1}}{2}y^2).$$

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If $g \in \mathrm{B}$ then:
$$g = \begin{pmatrix} a & 0 \\ r & a^{-1} \end{pmatrix}.$$

In this case we have:
$$\begin{aligned} a_g &= \sigma(a), \\ R_g(x, y) &= \frac{-ra^{-1}}{2} x^2 \cdot \delta_{y=a^{-1}x}. \end{aligned}$$

Altogether we have:
$$K_\rho(g, x, y) = a_g \cdot \psi(\frac{-ra^{-1}}{2} x^2) \delta_{y=a^{-1}x}. \tag{A.3.1}$$

A.3.3 Formulas for the Heisenberg representation

On $\mathcal{H}$ we also have a representation of the Heisenberg group H. We denote it by $\pi : \mathrm{H} \longrightarrow \mathrm{GL}(\mathcal{H})$. For every element $h \in \mathrm{H}$ we have a kernel $K_h : \mathbb{F}_q^2 \longrightarrow \mathbb{C}$. We denote by $K_\pi : \mathrm{H} \times \mathbb{F}_q^2 \longrightarrow \mathbb{C}$ the function of kernels. For an element $h \in \mathrm{H}$ the kernel $K_\pi(h)$ has the form $\psi(R_h(x, y))$ where R_h is an affine function which is supported on a certain one dimensional subspace of $\mathbb{F}_q^2$. Here are the exact formulas:

For an element $h = (\mathrm{q}, \mathrm{p}, \lambda)$ we have:
$$R_h(x, y) = (\frac{\mathrm{pq}}{2} + \mathrm{p}x + \lambda) \delta_{y=x+\mathrm{q}},$$

$$K_\pi(h, x, y) = \psi(\frac{\mathrm{pq}}{2} + \mathrm{p}x + \lambda) \delta_{y=x+\mathrm{q}}. \tag{A.3.2}$$

A.3.4 Formulas for the representation of the Jacobi group

The representations $\rho : \mathrm{Sp} \longrightarrow \mathrm{GL}(\mathcal{H})$ and $\pi : \mathrm{H} \longrightarrow \mathrm{GL}(\mathcal{H})$ combine together to give a representation of the semi-direct product $\mathrm{G} = \mathrm{Sp} \ltimes \mathrm{H}$. The group G is sometimes referred in literature as the *Jacobi group*. We denote the total representation by $\rho \ltimes \pi : \mathrm{G} \longrightarrow \mathrm{GL}(\mathcal{H})$, $\rho \ltimes \pi(g, h) = \rho(g) \cdot \pi(h)$. The representation $\rho \ltimes \pi$ is given by a kernel $K_{\rho \ltimes \pi} : \mathrm{G} \times \mathbb{F}_q^2 \longrightarrow \mathbb{C}$. We denote this kernel simply by K.

We give an explicit formula for the kernel K only in the case where $(g, h) \in \mathrm{B}w\mathrm{B} \times \mathrm{H}$, i.e.,:
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $b \neq 0$ and $h = (\mathrm{q}, \mathrm{p}, \lambda)$. In this case:
$$\begin{aligned} \mathrm{R}(g, h, x, y) &= R_g(x, y - \mathrm{q}) + R_h(y - \mathrm{q}, y), & \text{(A.3.3)} \\ K(g, h, x, y) &= a_g \cdot \psi(R_g(x, y - \mathrm{q}) + R_h(y - \mathrm{q}, y)). & \text{(A.3.4)} \end{aligned}$$

A.4 Deligne's letter

In this section we *geometrize* (Theorem A.5) the total representation $\rho \ltimes \pi : \mathrm{G} \longrightarrow \mathcal{H}$. First, we realize all finite *sets* as rational points of certain algebraic *varieties*. Beginning with the vector space, we take $\mathrm{V} = \mathbf{V}(\mathbb{F}_q)$. Next we replace all groups. We take, $\mathrm{H} = \mathbf{H}(\mathbb{F}_q)$, where $\mathbf{H} = \mathbf{V} \times \mathbb{G}_a$, $\mathrm{Sp} = \mathbf{Sp}(\mathbb{F}_q)$ and finally $\mathrm{G} = \mathbf{G}(\mathbb{F}_q)$, where $\mathbf{G} = \mathbf{Sp} \times \mathbf{H}$. The second step is to replace the kernel $K = K_{\rho \ltimes \pi} : \mathrm{G} \times \mathbb{F}_q^2 \longrightarrow \mathbb{C}$ (see (A.3.4)) by a *sheaf* theoretic object. Recall that K is a kernel of a representation and hence satisfies multiplication property:

$$(m, \mathrm{Id})^* K = p_1^* K * p_2^* K, \tag{A.4.1}$$

where m denotes the multiplication map, and $m \times \mathrm{Id} : \mathrm{G} \times \mathrm{G} \times \mathbb{F}_q^2 \to \mathrm{G} \times \mathbb{F}_q^2$ is given by $(g_1, g_2, x, y) \mapsto (g_1 g_2, x, y)$. Finally, $p_i(g_1, g_2, x, y) = (g_i, x, y)$ are the projections on the first and second G-coordinate respectively. The r.h.s of (A.4.1) is the convolution:

$$p_1^* K * p_2^* K(g_1, g_2, x, y) = \sum_{z \in \mathbb{F}_q} K(g_1, x, z) \cdot K(g_2, z, y).$$

In the sequel, we will usually suppress the $\mathbb{F}_q^2$ coordinates, writing $m : \mathrm{G} \times \mathrm{G} \to \mathrm{G}$, and $p_i : \mathrm{G} \times \mathrm{G} \to \mathrm{G}$. Moreover, to enhance the clarity of the notation, we will also suppress the projections p_i in (A.4.1), yielding a much cleaner statement:

$$m^* K = K * K, \tag{A.4.2}$$

These conventions will continue to hold also in the geometric setting, which is exactly where we will proceed to. We replace the kernel K by Deligne's *Weil representation sheaf* [D1]. This is an object $\mathcal{K} \in \mathrm{D}_c^b(\mathbf{G} \times \mathbb{A}^2)$ that satisfies[13] the analogue (to (A.4.2)) multiplication property:

$$m^* \mathcal{K} \simeq \mathcal{K} * \mathcal{K},$$

and its function is:

$$f^{\mathcal{K}} = K.$$

A.4.1 Existence of the Weil representation sheaf

The strategy. The method of constructing the Weil representation sheaf $\mathcal{K}$ is reminiscent to some extent to the construction of an analytic function via an analytic continuation. In the realm of perverse sheaves one uses the operation of *perverse extension* (cf. [BBD]), or, maybe, preferably called in our context *middle extension*[14]. The main idea is to construct, using formulas, an explicit irreducible (shifted) perverse sheaf $\mathcal{K}_{\mathbf{O}}$ on a "good" open subvariety $\mathbf{O} \subset \mathbf{G} \times \mathbb{A}^2$

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[13]However, in this paper we will prove a weaker property (see Theorem A.5) which is sufficient for our purposes.

[14]We use this as a unified terminology for taking a perverse extension with respect to any chosen perverse t-structure

and then we obtain the sheaf $\mathcal{K}$ by perverse extension of $\mathcal{K}_{\mathbf{O}}$ to the whole variety $\mathbf{G} \times \mathbb{A}^2$.

Preliminaries. We use in our construction the identifications $(\mathbf{V}, \omega) \simeq (\mathbb{A}^2, \omega_{\mathrm{std}})$, and $\mathbf{Sp} \simeq \mathbf{SL_2}$. We denote by $\mathbf{O}$ the open subvariety:

$$\mathbf{O} = \mathbf{O}_w \times \mathbf{H} \times \mathbb{A}^2,$$

where $\mathbf{O}_w$ denotes the (opposite) big Bruhat cell $\mathbf{B}w\mathbf{B} \subset \mathbf{SL_2}$.

In the sequel we will frequently make use of the *character* property (cf. [Ga]) of the sheaves $\mathscr{L}_\psi$ and $\mathscr{L}_\sigma$, that is,:

$$s^*\mathscr{L}_\psi \quad \simeq \quad \mathscr{L}_\psi \boxtimes \mathscr{L}_\psi, \tag{A.4.3}$$
$$m^*\mathscr{L}_\sigma \quad \simeq \quad \mathscr{L}_\sigma \boxtimes \mathscr{L}_\sigma, \tag{A.4.4}$$

where $s : \mathbb{G}_a \times \mathbb{G}_a \longrightarrow \mathbb{G}_a$ and $m : \mathbb{G}_m \times \mathbb{G}_m \longrightarrow \mathbb{G}_m$ denotes the addition and multiplication morphisms correspondingly, and $\boxtimes$ means exterior tensor product of sheaves.

Finally, given a sheaf $\mathcal{L}$ we use the notation $\mathcal{L}[i]$ for the translation functors and the notation $\mathcal{L}(i)$ for the i'th Tate twist.

Construction of the sheaf $\mathcal{K}$. In the first step we sheafify the kernel $K_{\rho \ltimes \pi}$ of the total representation, when restricted to the set $\mathrm{O} = \mathrm{O}_w \times \mathrm{H} \times \mathbb{F}_q^2$, using the formula (A.3.4). We obtain a sheaf $\mathcal{K}_{\mathbf{O}}$ on the open subvariety $\mathbf{O} = \mathbf{O}_w \times \mathbf{H} \times \mathbb{A}^2$:

$$\mathcal{K}_{\mathbf{O}} = \mathcal{A}_{\mathbf{O}} \otimes \tilde{\mathcal{K}}_{\mathbf{O}},$$

where $\tilde{\mathcal{K}}_{\mathbf{O}}$ is the sheaf of the non-normalized kernels and $\mathcal{A}_{\mathbf{O}}$ is the sheaf of the normalization coefficients. The sheaves $\tilde{\mathcal{K}}_{\mathbf{O}}$ and $\mathcal{A}_{\mathbf{O}}$ are constructed as follows. Define the morphism $\mathrm{R} :$ $\mathbf{O}_w \times \mathbf{H} \times \mathbb{A}^2 \longrightarrow \mathbb{A}^1$ by formula (A.3.3) and let $pr : \mathbf{O}_w \times \mathbf{H} \times \mathbb{A}^2 \longrightarrow \mathbf{O}_w$ be the projection morphism on the $\mathbf{O}_w$ coordinate. Now take:

$$\tilde{\mathcal{K}}_{\mathbf{O}} = \mathrm{R}^*\mathscr{L}_\psi,$$
$$\mathcal{A}_{\mathbf{O}} = pr^*\mathcal{A}_{\mathbf{O}_w},$$

where:

$$\mathcal{A}_{\mathbf{O}_w}(g) = \int_{x \in \mathbb{A}^1} \mathscr{L}_{\psi(\frac{1}{2}bx)} \otimes \mathscr{L}_{\sigma(x)}[2](1).$$

Here $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and the notation $\int_{x \in \mathbb{A}^1}$ means integration with compact support along the $\mathbb{A}^1$ fiber. It is a direct consequence from the construction:

Corollary A.4 *The sheaf $\mathcal{K}_{\mathbf{O}}$ is geometrically irreducible $[\dim(\mathbf{G}) + 1]$-perverse of pure weight zero and its function agrees with K on O, that is, $f^{\mathcal{K}_{\mathbf{O}}} = K_{|_O}$.*

Let j denote the open imbedding $j : \mathbf{O} \to \mathbf{G} \times \mathbb{A}^2$. We define the Weil representation sheaf $\mathcal{K}$ as the middle extension:

$$\mathcal{K} = j_{!*}\mathcal{K}_{\mathbf{O}}. \tag{A.4.5}$$

We are now ready to state the main theorem of this section:

Theorem A.5 (Weil representation sheaf [D1]) *The sheaf $\mathcal{K}$ is geometrically irreducible $[\dim(\mathbf{G}) + 1]$-perverse Weil sheaf on $\mathbf{G} \times \mathbb{A}^2$, of pure weight $w(\mathcal{K}) = 0$. The sheaf $\mathcal{K}$ satisfies the following properties:*

1. (Function) The function associated to $\mathcal{K}$, by sheaf to function correspondence, is the Weil representation kernel $f^{\mathcal{K}} = K$.

2. (Multiplication) For every element $g \in \mathbf{G}$ there exists isomorphisms:

$$\mathcal{K}_{|_g} * \mathcal{K} \simeq L_g^*(\mathcal{K}),$$

and:

$$\mathcal{K} * \mathcal{K}_{|_g} \simeq R_g^*(\mathcal{K}),$$

where $\mathcal{K}_{|_g}$ denotes the restriction of $\mathcal{K}$ to the closed subvariety $g \times \mathbb{A}^2$ and R_g, $L_g : \mathbf{G} \longrightarrow \mathbf{G}$ are the morphisms of right and left multiplication by g respectively.

For a proof, see Section A.5.

Remark. In Property 2. the notation $g \in \mathbf{G}$ means that g is a morphism $g : Spec(\overline{\mathbb{F}}_p) \to \mathbf{G}$. In fact a finer statement is true, namely, if $g \in \mathbf{G}(\mathbb{F}_q)$, $q = p^n$, then:

$$\mathcal{K}_{|_g} * \mathcal{K} \simeq L_g^*(\mathcal{K}),$$

and:

$$\mathcal{K} * \mathcal{K}_{|_g} \simeq R_g^*(\mathcal{K}),$$

are isomorphisms of Weil sheaves on $\mathbf{G} \times \mathbb{A}^2$, considered as an algebraic variety over $\mathbb{F}_q$.

A.5   Proofs

In this section we give the proofs for all technical facts that appeared in Part A of the Appendix.

Proof of Theorem A.3. Before giving the proof, we introduce a structure which is inherent to configurations of triple Lagrangian subspaces. Let $L_1, L_2, L_3 \subset V$ be three Lagrangian subspaces which are in a general position. In our case, these are just three different lines in the plane. Then the space L_j induces an isomorphism $r_{L_i, L_k} : L_k \longrightarrow L_i$, $i \neq j \neq k$, which is given by the rule $r_{L_i, L_k}(l_k) = l_i$, where $l_k + l_i \in L_j$.

The actual proof of the theorem will be given in two parts. In the first part we deal with the case where the three lines $L_1, L_2, L_2 \in \mathrm{Lag}$ are in a general position. In the second part we deal with the case when two of the three lines are equal to each other.

Part 1. (General position) Let $L_1^\circ, L_2^\circ, L_3^\circ \in \mathrm{Lag}^\circ$ be three oriented lines in a general position. Using the presentation (A.1.2) we can write:

$$\mathsf{F}_{L_3^\circ, L_2^\circ} \circ \mathsf{F}_{L_2^\circ, L_1^\circ} = \mathrm{a}_{L_3^\circ, L_2^\circ} \cdot \mathrm{a}_{L_2^\circ, L_1^\circ} \cdot \tilde{\mathsf{F}}_{L_3^\circ, L_2^\circ} \circ \tilde{\mathsf{F}}_{L_2^\circ, L_1^\circ},$$

$$\mathsf{F}_{L_3^\circ, L_1^\circ} = \mathrm{a}_{L_3^\circ, L_1^\circ} \cdot \tilde{\mathsf{F}}_{L_3^\circ, L_1^\circ}.$$

The result for Part 1 is a consequence of the following three simple lemmas (to be proved below):

Lemma A.6 *The following equality holds:*

$$\tilde{F}_{L_3^\circ, L_2^\circ} \circ \tilde{F}_{L_2^\circ, L_1^\circ} = \mathrm{C} \cdot \tilde{F}_{L_3^\circ, L_1^\circ},$$

where $\mathrm{C} = \sum_{l_2 \in \mathrm{L}_2} \psi(\frac{1}{2} \omega(l_2, \mathrm{r}_{\mathrm{L}_3, \mathrm{L}_2}(l_2)))$.

Lemma A.7 *The following equality holds:*

$$\mathrm{a}_{L_3^\circ, L_2^\circ} \cdot \mathrm{a}_{L_2^\circ, L_1^\circ} = \mathrm{D} \cdot \mathrm{a}_{L_3^\circ, L_1^\circ},$$

where $\mathrm{D} = \frac{1}{q} \sum_{l_2 \in \mathrm{L}_2} \psi(-\frac{1}{2} \omega(l_2, \mathrm{r}_{\mathrm{L}_3, \mathrm{L}_2}(\xi_{\mathrm{L}_2}))) \varrho_{\mathrm{L}_2}(l_2) \varrho_{\mathrm{L}_2}(\xi_{\mathrm{L}_2})$

Lemma A.8 *The following equality holds:*

$$\mathrm{D} \cdot \mathrm{C} = 1.$$

Part 2. (Non-general position) It is enough to check the following equalities:

$$\mathsf{F}_{L_1^\circ, L_2^\circ} \circ \mathsf{F}_{L_2^\circ, L_1^\circ} = \mathrm{I}, \tag{A.5.1}$$

$$\mathsf{F}_{L_1^{\bar{\sigma}}, L_2^\circ} \circ \mathsf{F}_{L_2^\circ, L_1^\circ} = -\mathrm{I}, \tag{A.5.2}$$

where $L_1^{\bar{\sigma}}$ has the opposite orientation to L_1°. We verify equation (A.5.1). The verification of (A.5.2) is done in the same way, therefore we omit it.

Write:

$$\tilde{\mathsf{F}}_{L_1^\circ, L_2^\circ}(\tilde{\mathsf{F}}_{L_2^\circ, L_1^\circ}(f))(h) = \sum_{l_1 \in \mathrm{L}_1} \sum_{l_2 \in \mathrm{L}_2} f(l_2 l_1 h), \tag{A.5.3}$$

where $f \in \mathcal{H}_{\mathrm{L}_1}$ and $h \in \mathrm{H}$. Both sides of (A.5.1) are self intertwining operators of $\mathcal{H}_{\mathrm{L}_1}$ and therefore they are proportional. Hence it is sufficient to compute (A.5.3) for a specific function

f and specific element $h \in \mathrm{H}$. We take $h = 0$ and $f = \delta_0$, where $\delta_0(\lambda l h) = \psi(\lambda)$ if $h = 0$ and equals 0 otherwise. We get:

$$\sum_{l_1 \in \mathrm{L}_1} \sum_{l_2 \in \mathrm{L}_2} f(l_2 l_1 h) = q.$$

Now write:

$$a_{\mathrm{L}_1^\circ, \mathrm{L}_2^\circ} \cdot a_{\mathrm{L}_2^\circ, \mathrm{L}_1^\circ} = \frac{1}{q^2} \sum_{l_1 \in \mathrm{L}_1,\, l_2 \in \mathrm{L}_2} \psi(\tfrac{1}{2}\omega(l_2, \xi_{\mathrm{L}_1}) + \tfrac{1}{2}\omega(l_1, \xi_{\mathrm{L}_2}))\varrho_{\mathrm{L}_2}(l_2)\varrho_{\mathrm{L}_1}(\xi_{\mathrm{L}_1})\varrho_{\mathrm{L}_1}(l_1)\varrho_{\mathrm{L}_2}(\xi_{\mathrm{L}_2}).$$

We identify L_2 and L_1 with the field $\mathbb{F}_q$ by the rules $s \cdot 1 \longmapsto s \cdot \xi_{\mathrm{L}_2}$ and $t \cdot 1 \longmapsto t \cdot \xi_{\mathrm{L}_1}$ correspondingly. In terms of these identifications we get:

$$a_{\mathrm{L}_1^\circ, \mathrm{L}_2^\circ} \cdot a_{\mathrm{L}_2^\circ, \mathrm{L}_1^\circ} = \frac{1}{q^2} \sum_{t,\, s \in \mathbb{F}_q} \psi(\tfrac{1}{2} s\omega(\xi_{\mathrm{L}_2}, \xi_{\mathrm{L}_1}) + \tfrac{1}{2} t\omega(\xi_{\mathrm{L}_1}, \xi_{\mathrm{L}_2}))\sigma(t)\sigma(s). \tag{A.5.4}$$

Denote by $a = \omega(\xi_{\mathrm{L}_2}, \xi_{\mathrm{L}_1})$. The right-hand side of (A.5.4) is equal to:

$$\frac{1}{q^2} \sum_{s \in \mathbb{F}_q} \psi(\tfrac{1}{2} as)\sigma(s) \cdot \sum_{t \in \mathbb{F}_q} \psi(-\tfrac{1}{2} at)\sigma(t) = \frac{q}{q^2} = \frac{1}{q}.$$

All together we get:

$$\mathsf{F}_{\mathrm{L}_1^\circ, \mathrm{L}_2^\circ} \circ \mathsf{F}_{\mathrm{L}_2^\circ, \mathrm{L}_1^\circ} = \mathrm{I}.$$

Proof of Lemma A.6. The proof is by direct computation. Write:

$$\tilde{\mathsf{F}}_{\mathrm{L}_3^\circ, \mathrm{L}_1^\circ}(f)(h) \;\; = \;\; \sum_{l_3 \in \mathrm{L}_3} f(l_3 h), \tag{A.5.5}$$

$$\tilde{\mathsf{F}}_{\mathrm{L}_3^\circ, \mathrm{L}_2^\circ}(\,\tilde{\mathsf{F}}_{\mathrm{L}_2^\circ, \mathrm{L}_1^\circ}(f)\,)(h) \;\; = \;\; \sum_{l_3 \in \mathrm{L}_3} \sum_{l_2 \in \mathrm{L}_2} f(l_2 l_3 h), \tag{A.5.6}$$

where $f \in \mathcal{H}_{\mathrm{L}_1}$ and $h \in \mathrm{H}$. Both (A.5.5) and (A.5.6) are intertwining operators from $\mathcal{H}_{\mathrm{L}_1}$ to $\mathcal{H}_{\mathrm{L}_3}$, therefore they are proportional. In order to compute the proportionality coefficient C it is enough to compute (A.5.5) and (A.5.6) for specific f and specific e. We take $h = 0$ and $f = \delta_0$ where $\delta_0(\mathrm{q}, \mathrm{p}, \lambda) = \psi(\lambda)$. We get:

$$\tilde{\mathsf{F}}_{\mathrm{L}_3^\circ, \mathrm{L}_1^\circ}(\delta_0) = 1,$$

$$\tilde{\mathsf{F}}_{\mathrm{L}_3^\circ, \mathrm{L}_2^\circ}(\,\tilde{\mathsf{F}}_{\mathrm{L}_2^\circ, \mathrm{L}_1^\circ}(\delta_0)\,)(0) = \sum_{l_2 + l_3 \in \mathrm{L}_1} \psi(\tfrac{1}{2}\omega(l_2, l_3)). \tag{A.5.7}$$

But the right-hand side of (A.5.7) is equal to:

$$\sum_{l_2 \in \mathrm{L}_2} \psi(\tfrac{1}{2}\omega(l_2, \mathrm{r}_{\mathrm{L}_3, \mathrm{L}_2}(l_2))).$$

■

Proof of Lemma A.7. The proof is by direct computation. Write:

$$a_{L_3^\circ, L_1^\circ} = \frac{1}{q} \sum_{l_1 \in L_1} \psi(\tfrac{1}{2}\omega(l_1, \xi_{L_3})) \varrho_{L_1}(l_1)\varrho_{L_2}(\xi_{L_3}),$$

$$a_{L_3^\circ, L_2^\circ} \cdot a_{L_2^\circ, L_1^\circ} = \frac{1}{q^2} \sum_{l_1 \in L_1,\, l_2 \in L_2} \psi(\tfrac{1}{2}\omega(l_1, \xi_{L_2}) + \tfrac{1}{2}\omega(l_2, \xi_{L_3}))\varrho_{L_1}(l_1)\varrho_{L_2}(\xi_{L_2})\varrho_{L_2}(l_2)\varrho_{L_3}(\xi_{L_3}).$$

(A.5.8)

The term $\psi(\tfrac{1}{2}\omega(l_1, \xi_{L_2}) + \tfrac{1}{2}\omega(l_2, \xi_{L_3}))$ is equal to:

$$\psi(\tfrac{1}{2}\omega(l_1, \xi_{L_2} - \xi_{L_3}) + \tfrac{1}{2}\omega(l_2, \xi_{L_3})) \cdot \psi(\tfrac{1}{2}\omega(l_1, \xi_{L_3})). \qquad (A.5.9)$$

We are free to choose ξ_{L_3} such that $\xi_{L_2} - \xi_{L_3} \in L_1$. Therefore, using (A.5.9) we get that the right-hand side of (A.5.8) is equal to:

$$\frac{1}{q} \sum_{l_2 \in L_2} \psi(\tfrac{1}{2}\omega(l_2, \xi_{L_3}))\varrho_{L_2}(\xi_{L_2})\varrho_{L_2}(l_2) \cdot a_{L_3^\circ, L_1^\circ}. \qquad (A.5.10)$$

Now, substituting $\xi_{L_3} = -r_{L_3, L_2}(\xi_{L_2})$ in (A.5.10) we obtain:

$$\frac{1}{q} \sum_{l_2 \in L_2} \psi(-\tfrac{1}{2}\omega(l_2, r_{L_3, L_2}(\xi_{L_2})))\varrho_{L_2}(l_2)\varrho_{L_2}(\xi_{L_2}) \cdot a_{L_3^\circ, L_1^\circ}.$$

■

Proof of Lemma A.8. Identify L_2 with $\mathbb{F}_q$ by the rule $t \cdot 1 \longmapsto t \cdot \xi_{L_2}$. In terms of this identification we get:

$$D = \frac{1}{q} \sum_{t \in \mathbb{F}_q} \psi(-\tfrac{1}{2}\omega(t\xi_{L_2}, r_{L_3, L_2}(\xi_{L_2})))\varrho(t),$$

$$C = \sum_{t \in \mathbb{F}_q} \psi(\tfrac{1}{2}\omega(t\xi_{L_2}, r_{L_3, L_2}(t\xi_{L_2}))).$$

Denote by $a = \omega(\xi_{L_2}, r_{L_3, L_2}(\xi_{L_2}))$. Then:

$$D = \frac{1}{q} \sum_{t \in \mathbb{F}_q} \psi(-\tfrac{1}{2}at)\varrho(t),$$

$$C = \sum_{t \in \mathbb{F}_q} \psi(\tfrac{1}{2}at^2).$$

Now, we have the following remarkable equality:

$$\sum_{t \in \mathbb{F}_q} \psi(\tfrac{1}{2}at)\varrho(t) = \sum_{t \in \mathbb{F}_q} \psi(\tfrac{1}{2}at^2).$$

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This, combined with $C \cdot \overline{C} = q$, gives the result. ∎

This completes the proof of Part 2 and of Theorem A.3 ∎

Proof of Theorem A.5. The sheaf $\mathcal{K}$ is obtained by middle extension from the open subvariety **O** of the sheaf $\mathcal{K}_{\mathbf{O}}$. The sheaf $\mathcal{K}_{\mathbf{O}}$ is clearly geometrically irreducible $[\dim(\mathbf{G}) + 1]$-perverse of pure weight 0. This implies that the sheaf $\mathcal{K}$ is also geometrically irreducible $[\dim(\mathbf{G}) + 1]$-perverse of pure weight $w(\mathcal{K}) = 0$.

Proof of Property 1. Assuming the validity of Property 2, we prove Property 1. Restricting the sheaf $\mathcal{K}$ to the open subvariety **O**, and taking sheaf to function correspondence, one obtains $f^{\mathcal{K}} = f^{\mathcal{K}_{\mathbf{O}}} = K_{|\mathbf{O}}$. Applying sheaf to function correspondence to the multiplication isomorphism, we get that $f^{\mathcal{K}}$ is multiplicative. Finally, we use the fact that the set $\mathrm{O}_w \times \mathrm{H}$ is a generating set of G. Therefore, we have two functions K, and $f^{\mathcal{K}}$ that coincide on a generating set and satisfy multiplication property. This implies that they must coincide on the whole domain.

Proof of Property 2. What remains is to prove the multiplication property, namely, Property 2. We will prove the left multiplication isomorphism. The proof of the right multiplication isomorphism follows the same lines, therefore we omit it. In the course of the proof we shall use the following auxiliary sheaves:

• We sheafify the kernel K_π using the formula (A.3.2) and obtain a sheaf on $\mathbf{H} \times \mathbb{A}^2$:

$$\mathcal{K}_\pi(h, x, y) = \mathscr{L}_{\psi(\frac{1}{2}\,\mathrm{pq} + \mathrm{p}x + \lambda)} \otimes \delta_{y=x+\mathrm{q}}.$$

Where $h = (\mathrm{q}, \mathrm{p}, \lambda)$, and we use the notation δ for the skyscraper sheaf.

• We sheafify the kernel K_ρ when restricted to the set $\mathrm{B} \times \mathbb{F}_q^2$ using the formula (A.3.1) and obtain a sheaf on the variety $\mathbf{B} \times \mathbb{A}^2$, which we denote by $\mathcal{K}_{\mathbf{B}}$,

$$\mathcal{K}_{\mathbf{B}}(b, x, y) = \mathscr{L}_{\sigma(a)} \otimes \mathscr{L}_{\psi(\frac{1}{2}\,ra^{-1}x^2)} \otimes \delta_{y=a^{-1}x}.$$

Where $b = \left(\begin{smallmatrix} a & 0 \\ r & a^{-1} \end{smallmatrix}\right)$.

• We will frequently make use of several other sheaves obtained by restrictions from $\mathcal{K}_{\mathbf{O}}$, $\mathcal{A}_{\mathbf{O}}$, $\mathcal{K}_{\mathbf{B}}$ and $\mathcal{A}_{\mathbf{B}}$. Suppose $\mathbf{X} \subset \mathbf{O}_w \times \mathbf{H}$ is a subvariety. Then we define $\mathcal{K}_{\mathbf{X}} = \mathcal{K}_{\mathbf{O}|_{\mathbf{X} \times \mathbb{A}^2}}$ and $\mathcal{A}_{\mathbf{X}} = \mathcal{A}_{\mathbf{O}|_{\mathbf{X} \times \mathbb{A}^2}}$. The same when $\mathbf{X} \subset \mathbf{B}$. Finally, we denote by δ_0 the *sky-scraper* sheaf on $\mathbb{A}^1$ which corresponds to the delta function at zero.

The proof of Property 2 will proceed in several steps:

Step 1. It is sufficient to prove the multiplication property separately for the Weyl element w, an element $b \in \mathbf{B}$ and an element $h \in \mathbf{H}$. This follows from the Bruhat decomposition, Corollary A.12 below and the following decomposition lemma:

Lemma A.9 *There exists isomorphisms:*

$$\mathcal{K}_{\mathbf{O}} \simeq \mathcal{K}_{\mathbf{B}} * \mathcal{K}_w * \mathcal{K}_{\mathbf{U}} * \mathcal{K}_\pi,$$

where $\mathbf{U}$ *denote the unipotent radical of* $\mathbf{B}$ *and* w *is the Weyl element.*

Step 2. We prove Property 2 for the Weyl element, $g = w$. We want to construct an isomorphism:

$$\mathcal{K}_{|_w} * \mathcal{K} \simeq L_w^* \mathcal{K}. \tag{A.5.11}$$

Both sides of (A.5.11) are irreducible $[\dim(\mathbf{G})+1]$-perverse, therefore it is sufficient to construct an isomorphism on the open subvariety $\mathbf{O}^\times = \mathbf{O} \cap w\mathbf{O}$. This has the advantage that over $\mathbf{O}^\times$ we have formulas for $\mathcal{K}$, and moreover, L_w maps $\mathbf{O}^\times$ into itself. We consider two decompositions:

$$\begin{aligned}
\mathbf{O}^\times &\simeq \mathbf{U}^{\circ\times} \times \mathbf{B} \times \mathbf{H}, \\
\mathbf{O}^\times &\simeq \mathbf{U}^\times w \times \mathbf{B} \times \mathbf{H},
\end{aligned} \tag{A.5.12}$$

where $\mathbf{U}^\times = \mathbf{U} \smallsetminus \{\mathbf{I}\}$ and $\mathbf{U}^\circ$ denotes the standard unipotent radical (Recall that in our convention $\mathbf{U}$ denotes the group of unipotent lower triangular matrices) . In terms of the above decompositions we have the following isomorphisms:

Claim A.10 *There exist isomorphisms:*

1. $\mathcal{K}_{\mathcal{U}}(u^\circ bh) \simeq \mathcal{K}_{\mathbf{U}^{\circ\times}}(u^\circ) * \mathcal{K}_{\mathbf{B}}(b) * \mathcal{K}_\pi(h)$.

2. $\mathcal{K}_{\mathcal{U}}(uwbh) \simeq \mathcal{K}_{\mathbf{U}^\times w}(uw) * \mathcal{K}_{\mathbf{B}}(b) * \mathcal{K}_\pi(h)$.

Now, restricting to $\mathbf{O}^\times$ and using the decomposition (A.5.12) our *main statement* is the existence of an isomorphism:

$$\mathcal{K}_w * \mathcal{K}_{\mathcal{U}}(u^\circ bh) \simeq \mathcal{K}_{\mathcal{U}}(wu^\circ bh). \tag{A.5.13}$$

Indeed, on developing the right-hand side of (A.5.13) we obtain:

$$\begin{aligned}
\mathcal{K}_{\mathcal{U}}(wu^\circ bh) &= \mathcal{K}_{\mathcal{U}}(u^{\circ w} wbh) \\
&\simeq \mathcal{K}_{\mathbf{U}^\times w}(u^{\circ w} w) * \mathcal{K}_{\mathbf{B}}(b) * \mathcal{K}_\pi(h) \\
&\simeq \mathcal{K}_w * (\mathcal{K}_{\mathbf{U}^{\circ\times}}(u^\circ) * \mathcal{K}_{\mathbf{B}}(b) * \mathcal{K}_\pi(h)) \\
&\simeq \mathcal{K}_w * \mathcal{K}_{\mathcal{U}}(u^\circ bh),
\end{aligned}$$

where $u^{\circ w} = wu^\circ w^{-1}$. The first and third isomorphisms are applications of Claim A.10 parts 2 and 1 respectively. The second isomorphism is a result of associativity of convolution and the following *central lemma*:

Lemma A.11 *There exists an isomorphism:*

$$\mathcal{K}_{\mathbf{U}^\times w}(u^{\circ w} w) \simeq \mathcal{K}_w * \mathcal{K}_{\mathbf{U}^{\circ\times}}(u^\circ). \tag{A.5.14}$$

The following is a consequence of (A.5.11).

Corollary A.12 *There exists an isomorphism:*

$$\mathcal{K}_{|\mathbf{B}\times\mathbf{H}} \simeq \mathcal{K}_{\mathbf{B}} * \mathcal{K}_{\pi}. \tag{A.5.15}$$

Proof. On developing the left-hand side of (A.5.15) we obtain:

$$
\begin{aligned}
\mathcal{K}_{|\mathbf{B}\times\mathbf{H}} &\simeq \mathcal{K}_w * \mathcal{K}_{|w^{-1}\mathbf{B}\times\mathbf{H}} \\
&\simeq \mathcal{K}_w * (\mathcal{K}_{w^{-1}} * \mathcal{K}_{\mathbf{B}} * \mathcal{K}_{\pi}) \\
&\simeq (\mathcal{K}_w * \mathcal{K}_{w^{-1}}) * \mathcal{K}_{\mathbf{B}} * \mathcal{K}_{\pi} \\
&\simeq \mathcal{K}_{\mathbf{B}} * \mathcal{K}_{\pi}.
\end{aligned}
$$

The first isomorphism is a consequence of (A.5.11). The second isomorphism is a consequence of Lemma A.9. The third isomorphism is the associativity property of convolution. The last isomorphism is a property of the Fourier transform [KL], that is, $\mathcal{K}_w * \mathcal{K}_{w^{-1}} \simeq \mathcal{I}$, where $\mathcal{I}$ is the kernel of the identity operator. ∎

Step 3. We prove Property 2 for element $b \in \mathbf{B}$. Using corollary A.12 we have $\mathcal{K}_{|_b} \simeq \mathcal{K}_{\mathbf{B}|_b} = \mathcal{K}_b$. We want to construct an isomorphism:

$$\mathcal{K}_b * \mathcal{K} \simeq L_b^* \mathcal{K}. \tag{A.5.16}$$

Since both sides of (A.5.16) are irreducible (shifted) perverse sheaves, it is enough to construct an isomorphism on the open set $\mathbf{O} = \mathbf{O}_w \times \mathbf{H} \times \mathbb{A}^2$. Write:

$$\mathcal{K}_b * \mathcal{K}_{|\mathbf{O}} \simeq \mathcal{K}_b * \mathcal{K}_{\mathbf{O}} \simeq \mathcal{K}_b * (\mathcal{K}_{\mathbf{B}} * \mathcal{K}_w * \mathcal{K}_{\mathbf{U}} * \mathcal{K}_{\pi}) \simeq (\mathcal{K}_b * \mathcal{K}_{\mathbf{B}}) * (\mathcal{K}_w * \mathcal{K}_{\mathbf{U}} * \mathcal{K}_{\pi}).$$

The first isomorphism is by construction. The second isomorphism is an application of Lemma A.9. The third isomorphism is the associativity property of the convolution operation between sheaves. From the last isomorphism we see that it is enough to construct an isomorphism $\mathcal{K}_b * \mathcal{K}_{\mathbf{B}} \simeq L_b^*(\mathcal{K}_{\mathbf{B}})$, where $L_b : \mathbf{B} \longrightarrow \mathbf{B}$. The construction is an easy consequence of formula (A.3.1) and the character sheaf property (A.4.3) of $\mathscr{L}_{\psi}$.

Step 4. We prove Property 2 for an element $h \in \mathbf{H}$. We want to construct an isomorphism:

$$\mathcal{K}_{|_h} * \mathcal{K} \simeq L_h^* \mathcal{K}. \tag{A.5.17}$$

Both sides of (A.5.17) are irreducible $[\dim(\mathbf{G}) + 1]$-perverse. Therefore, it is sufficient to construct an isomorphism on the open set $\mathbf{O}$. This is done by a direct computation, very similar to what has been done before, hence we omit it. This concludes the proof of Theorem A.5. ∎

Proof of Lemma A.9. We will prove the lemma in two steps.

Step 1. We prove that $\mathcal{K}_{\mathbf{O}} \simeq \mathcal{K}_{\mathbf{O}_w} * \mathcal{K}_{\pi}$. In a more explicit form we want to show:

$$\mathcal{A}_{\mathbf{O}} \otimes \tilde{\mathcal{K}}_{\mathbf{O}} \simeq \mathcal{A}_{\mathbf{O}_w} \otimes \tilde{\mathcal{K}}_{\mathbf{O}_w} * \tilde{\mathcal{K}}_{\mathbf{H}}. \tag{A.5.18}$$

40

It is sufficient to show the existence of an isomorphism $\tilde{\mathcal{K}}_{\mathbf{O}} \simeq \tilde{\mathcal{K}}_{\mathbf{O}_w} * \tilde{\mathcal{K}}_{\mathbf{H}}$. On developing the left-hand side of (A.5.18) we obtain:

$$\tilde{\mathcal{K}}_{\mathbf{O}}(g, h, x, y) = \mathscr{L}_{\psi(R(g,h,x,y))}.$$

On developing the right-hand side we obtain:

$$
\begin{aligned}
\tilde{\mathcal{K}}_{\mathbf{O}_w} * \tilde{\mathcal{K}}_{\mathbf{H}}(\,(g, h)\,, x, y) &= \int_{z \in \mathbb{A}^1} \tilde{\mathcal{K}}_{\mathbf{O}_w}(g, x, z) \otimes \tilde{\mathcal{K}}_{\mathbf{H}}(h, z, y) \\
&= \int_{\mathbb{A}^1} \mathscr{L}_{\psi(R_g(x,z))} \otimes \mathscr{L}_{\psi(R_h(z,y))} \otimes \delta_{y=z-\mathsf{q}} \\
&\simeq \mathscr{L}_{\psi(R_g(x,y-\mathsf{q}))} \otimes \mathscr{L}_{\psi(R_h(y-\mathsf{q},y))} \\
&\simeq \mathscr{L}_{\psi(R_g(x,y-\mathsf{q})+R_h(y-\mathsf{q},y))} \\
&= \mathscr{L}_{\psi(R(g,h,x,y))}.
\end{aligned}
$$

The only non-trivial isomorphism is the last one and it is a consequence of the Artin-Schreier sheaf being a character sheaf on the additive group $\mathbb{G}_a$.

Step 2. We prove that $\mathcal{K}_{\mathbf{O}_w} \simeq \mathcal{K}_{\mathbf{B}} * \mathcal{K}_w * \mathcal{K}_{\mathbf{U}}$. In a more explicit form we want to show:

$$\mathcal{A}_{\mathbf{O}_w} \otimes \tilde{\mathcal{K}}_{\mathbf{O}_w} \simeq \mathcal{A}_{\mathbf{B}} \otimes \mathcal{A}_w \otimes \mathcal{A}_{\mathbf{U}} \otimes \tilde{\mathcal{K}}_{\mathbf{B}} * \tilde{\mathcal{K}}_w * \tilde{\mathcal{K}}_{\mathbf{U}}.$$

We will separately show the existence of two isomorphisms:

$$
\begin{aligned}
\tilde{\mathcal{K}}_{\mathbf{O}_w} &\simeq \tilde{\mathcal{K}}_{\mathbf{B}} * \tilde{\mathcal{K}}_w * \tilde{\mathcal{K}}_{\mathbf{U}}, && \text{(A.5.19)} \\
\mathcal{A}_{\mathbf{O}_w} &\simeq \mathcal{A}_{\mathbf{B}} \otimes \mathcal{A}_w \otimes \mathcal{A}_{\mathbf{U}}. && \text{(A.5.20)}
\end{aligned}
$$

Isomorphism (A.5.19). We have the decomposition, $\mathbf{O}_w \simeq \mathbf{B} \times w \times \mathbf{U}$. Let $b = \left(\begin{smallmatrix} a & 0 \\ r & a^{-1} \end{smallmatrix}\right)$ and $u = \left(\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right)$ be general elements in the groups $\mathbf{U}$ and $\mathbf{B}$ respectively. In terms of the coordinates (a, r, s) a general element in $\mathbf{O}_w$ is of the form $g = \left(\begin{smallmatrix} as & a \\ rs-a^{-1} & r \end{smallmatrix}\right)$. Developing the left-hand side of (A.5.19) in terms of the coordinates (a, r, s) we obtain:

$$\tilde{\mathcal{K}}_{\mathbf{O}_w}(bwu, x, y) = \mathscr{L}_{\psi(-\frac{1}{2} a^{-1}rx^2 + a^{-1}xy - \frac{1}{2} sy^2)}.$$

On developing the right-hand side of (A.5.19) we obtain:

$$
\begin{aligned}
\tilde{\mathcal{K}}_{\mathbf{B}} * \tilde{\mathcal{K}}_w * \tilde{\mathcal{K}}_{\mathbf{U}}(bwu, x, y) &= \int_{z,z' \in \mathbb{A}^1} \tilde{\mathcal{K}}_{\mathbf{B}}(b, x, z) \otimes \tilde{\mathcal{K}}_w(w, z, z') \otimes \tilde{\mathcal{K}}_{\mathbf{U}}(u, z', y) \\
&= \int_{z,z' \in \mathbb{A}^1} \mathscr{L}_{\psi(-\frac{1}{2} ra^{-1}x^2)} \otimes \delta_{x=az} \otimes \mathscr{L}_{\psi(zz')} \otimes \mathscr{L}_{\psi(-\frac{1}{2} sz'^2)} \otimes \delta_{y=z'} \\
&\simeq \mathscr{L}_{\psi(-\frac{1}{2} ra^{-1}x^2)} \otimes \mathscr{L}_{\psi(a^{-1}xy)} \otimes \mathscr{L}_{\psi(-\frac{1}{2} sy^2)} \\
&\simeq \mathscr{L}_{\psi(-\frac{1}{2} ra^{-1}x^2 + a^{-1}xy - \frac{1}{2} sy^2)}.
\end{aligned}
$$

The last isomorphism is a consequence of the fact that the Artin-Schreier sheaf is a character sheaf (A.4.3). Altogether we obtained isomorphism (A.5.19).

Isomorphism (A.5.20). On developing the left-hand side of (A.5.20) in terms of the co-ordinates (a, r, s) we obtain:

$$
\begin{aligned}
\mathcal{A}_{\mathbf{O}_w}(bwu) &= \mathcal{G}(\psi_a, \sigma)[2](-1) \\
&\simeq \mathcal{G}(\psi, \sigma_{a^{-1}})[2](-1) \\
&\simeq \mathscr{L}_{\sigma(a^{-1})} \otimes \mathcal{G}(\psi, \sigma)[2](-1) \\
&\simeq \mathscr{L}_{\sigma(a)} \otimes \mathcal{G}(\psi, \sigma)[2](-1) \\
&= \mathcal{A}_{\mathbf{B}} \otimes \mathcal{A}_w \otimes \mathcal{A}_{\mathbf{U}}(bwu),
\end{aligned}
$$

where $\mathcal{G}(\psi_s, \sigma_a) = \int_{\mathbb{A}^1} \mathscr{L}_{\psi(\frac{1}{2} sz)} \otimes \mathscr{L}_{\sigma(az)}$ denotes the quadratic *Gauss-sum* sheaf. The second isomorphism is a change of coordinates $z \mapsto az$ under the integration. The third isomorphism is a consequence of the Kummer sheaf $\mathscr{L}_\sigma$ being a character sheaf on the multiplicative group $\mathbb{G}_m$ (A.4.4). The fourth isomorphism is a specific property of the Kummer sheaf which is associated to the quadratic character σ. This completes the construction of isomorphism (A.5.20). ■

Proof of claim A.10. Carried out in exactly the same way as the proof of the decomposition Lemma A.9. Namely, using the explicit formulas of the sheaves $\mathcal{K}_{\mathbf{O}}, \mathcal{K}_{\mathbf{B}}, \mathcal{K}_\pi$ and the character sheaf property of the sheaves $\mathscr{L}_\psi$ and $\mathscr{L}_\sigma$. ■

Proof of Lemma A.11. First, we write isomorphism (A.5.14) in a more explicit form:

$$
\mathcal{A}_{\mathbf{U} \times \mathbf{w}} \otimes \tilde{\mathcal{K}}_{\mathbf{U} \times \mathbf{w}}(u^{\circ w} w) \simeq \mathcal{A}_w \otimes \mathcal{A}_{\mathbf{U}^{\circ \times}} \otimes \tilde{\mathcal{K}}_w * \tilde{\mathcal{K}}_{\mathbf{U}^{\circ \times}}(u^\circ). \tag{A.5.21}
$$

Let $u^\circ = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix} \right) \in \mathbf{U}^{\circ \times}$ be a non-trivial unipotent. Then $u^{\circ w} w = w u^\circ = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -s \end{smallmatrix} \right)$. On developing the left-hand side of (A.5.21) we obtain:

$$
\begin{aligned}
\tilde{\mathcal{K}}_{\mathbf{U} \times \mathbf{w}}(u^{\circ w} w, x, y) &= \mathscr{L}_{\psi(\frac{1}{2} sx^2 + xy)}, \\
\mathcal{A}_{\mathbf{U} \times \mathbf{w}}(u^{\circ w} w) &= \mathcal{G}(\psi, \sigma)[2](1),
\end{aligned}
$$

where $\mathcal{G}(\psi_s, \sigma_a) = \int_{\mathbb{A}^1} \mathscr{L}_{\psi(\frac{1}{2} sz)} \otimes \mathscr{L}_{\sigma(az)}$.

On developing the right-hand side of (A.5.21) we obtain:

$$
\begin{aligned}
\tilde{\mathcal{K}}_w * \tilde{\mathcal{K}}_{\mathbf{U}^{\circ \times}}(u^\circ, x, y) &= \int_{z \in \mathbb{A}^1} \tilde{\mathcal{K}}_w(x, z) \otimes \tilde{\mathcal{K}}_{\mathbf{U}^{\circ \times}}(u, z, y) \\
&= \int_{\mathbb{A}^1} \mathscr{L}_{\psi(xz)} \otimes \mathscr{L}_{\psi(-\frac{1}{2} s^{-1} z^2 + s^{-1} zy - \frac{1}{2} s^{-1} y^2)} \\
&\simeq \int_{\mathbb{A}^1} \mathscr{L}_{\psi(xz - \frac{1}{2} s^{-1} z^2 + s^{-1} zy - \frac{1}{2} s^{-1} y^2)} \\
&\simeq \int_{\mathbb{A}^1} \mathscr{L}_{\psi(-\frac{1}{2} s^{-1}(z - sx - y)^2)} \otimes \mathscr{L}_{\psi(\frac{1}{2} sx^2 + xy)} \\
&\simeq \int_{\mathbb{A}^1} \mathscr{L}_{\psi(-\frac{1}{2} s^{-1} z^2)} \otimes \mathscr{L}_{\psi(\frac{1}{2} sx^2 + xy)}.
\end{aligned}
$$

By applying change of coordinates $z \mapsto sz$ under the last integration we obtain:

$$
\int_{\mathbb{A}^1} \mathscr{L}_{\psi(-\frac{1}{2} s^{-1} z^2)} \otimes \mathscr{L}_{\psi(\frac{1}{2} sx^2 + xy)} \simeq \int_{z \in \mathbb{A}^1} \mathscr{L}_{\psi(-\frac{1}{2} sz^2)} \otimes \mathscr{L}_{\psi(\frac{1}{2} sx^2 + xy)}. \tag{A.5.22}
$$

Now write:
$$\mathcal{A}_w \otimes \mathcal{A}_{\mathbf{U}^{\circ \times}}(u^{\circ}) = \mathcal{G}(\psi, \sigma)[2](1) \otimes \mathcal{G}(\psi_s, \sigma)[2](1). \tag{A.5.23}$$

Combining (A.5.22) and (A.5.23) we obtain that the right-hand side of (A.5.21) is isomorphic to:
$$\left(\mathcal{G}(\psi_s, \sigma)[2](1) \otimes \int_{\mathbb{A}^1} \mathscr{L}_{\psi(-\frac{1}{2} sz^2)} \right) \otimes \left(\mathcal{G}(\psi, \sigma)[2](1) \otimes \mathscr{L}_{\psi(\frac{1}{2} sx^2 + xy)} \right).$$

The main argument is the existence of the following isomorphism:
$$\mathcal{G}(\psi_s, \sigma)[2](1) \otimes \int_{\mathbb{A}^1} \mathscr{L}_{\psi(-\frac{1}{2} sz^2)} \simeq \overline{\mathbb{Q}}_\ell.$$

It is a direct consequence of the following lemma:

Lemma A.13 (Main lemma) *There exists a canonical isomorphism of sheaves on* $\mathbb{G}_m$:
$$\int_{\mathbb{A}^1} \mathscr{L}_{\psi(\frac{1}{2} sz)} \otimes \mathscr{L}_{\sigma(z)} \simeq \int_{\mathbb{A}^1} \mathscr{L}_{\psi(\frac{1}{2} sz^2)},$$

where $s \in \mathbb{G}_m$.

Proof. The parameter s does not play any essential role in the argument. Therefore, it is sufficient to prove:
$$\int_{\mathbb{A}^1} \mathscr{L}_{\psi(z)} \otimes \mathscr{L}_{\sigma(z)} \simeq \int_{\mathbb{A}^1} \mathscr{L}_{\psi(z^2)}. \tag{A.5.24}$$

Define the morphism $p : \mathbb{G}_m \longrightarrow \mathbb{G}_m$, $p(x) = x^2$. The morphism p is an ètale double cover. We have $p_* \overline{\mathbb{Q}}_\ell \simeq \mathscr{L}_\sigma \oplus \overline{\mathbb{Q}}_\ell$. Now on developing the left-hand-side of (A.5.24) we obtain:
$$\int_{\mathbb{A}^1} \mathscr{L}_{\psi(z)} \otimes \mathscr{L}_{\sigma(z)} = \pi_!(\mathscr{L}_\psi \otimes \mathscr{L}_\sigma) \simeq \pi_!(\mathscr{L}_\psi \otimes (\mathscr{L}_\sigma \oplus \overline{\mathbb{Q}}_\ell)).$$

The first step is just a translation to conventional notations, where π stands for the projection $\pi : \mathbb{G}_m \longrightarrow pt$. The second isomorphism uses the fact that $\pi_! \mathscr{L}_\psi \simeq 0$. Next:
$$\pi_!(\mathscr{L}_\psi \otimes (\mathscr{L}_\sigma \oplus \overline{\mathbb{Q}}_\ell)) \simeq \pi_!(\mathscr{L}_\psi \otimes p_* \overline{\mathbb{Q}}_\ell) \simeq \pi_! p^* \mathscr{L}_\psi = \int_{\mathbb{A}^1} \mathscr{L}_{\psi(z^2)}. \quad \blacksquare$$
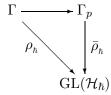
This completes the proof of proposition A.11. ∎

B   Proofs Section

For the remainder of this section we fix the following notations. Let $\hbar = \frac{1}{p}$, where p is a fixed prime $\neq 2, 3$. Consider the lattice Λ^* of characters of the torus $\mathbb{T}$ and the quotient vector space $V = \Lambda^*/p\Lambda^*$. The integral symplectic form on Λ^* is specialized to give a symplectic form on V, i.e., $\omega : V \times V \longrightarrow \mathbb{F}_p$. Fix $\psi : \mathbb{F}_p \longrightarrow \mathbb{C}^*$ a non-trivial additive character. Let $\mathcal{A}_\hbar$ be "the algebra of functions on the quantum torus" and $\Gamma \simeq SL_2(\mathbb{Z})$ its group of symmetries.

B.1 Proof of Theorem 2.2

Basic set-up: let $(\pi_\hbar, \mathcal{H}_\hbar)$ be a representation of $\mathcal{A}_\hbar$, which is a representative of the unique irreducible class which is fixed by Γ (cf. Theorem 2.1). Let $\rho_\hbar : \Gamma \longrightarrow \mathrm{PGL}(\mathcal{H}_\hbar)$ be the associated projective representation. Here we give a proof that $\rho_\hbar$ can be linearized in a unique way which factors through the quotient group $\Gamma_p \simeq \mathrm{SL}_2(\mathbb{F}_p)$:

$$
\begin{array}{ccc}
\Gamma & \longrightarrow & \Gamma_p \\
& \rho_\hbar \searrow & \downarrow \bar{\rho}_\hbar \\
& & \mathrm{GL}(\mathcal{H}_\hbar)
\end{array}
$$

The proof will be given in several steps.

Step 1. *Uniqueness.* The uniqueness of the linearization follows directly from the fact that the group $\mathrm{SL}_2(\mathbb{F}_p)$, $p \neq 2, 3$, has no characters.

Step 2. *Existence.*[15] The main technical tool in the proof of existence is a construction of a finite dimensional quotient of the algebra $\mathcal{A}_\hbar$. Let $\mathcal{A}_p$ be the algebra generated over $\mathbb{C}$ by the symbols $\{s(v) \mid v \in \mathrm{V}\}$ and quotient out by the relations:

$$
s(u + v) = \psi(\tfrac{1}{2}\,\omega(u, v))s(u)s(v). \tag{B.1.1}
$$

The algebra $\mathcal{A}_p$ is non-trivial and the vector space V gives on it a standard basis. These facts will be proven in the sequel. We have the following map:

$$
s : \mathrm{V} \longrightarrow \mathcal{A}_p, \ v \mapsto s(v).
$$

The group Γ_p acts on $\mathcal{A}_p$ by automorphisms through its action on V. We have a homomorphism of algebras:

$$
q : \mathcal{A}_\hbar \longrightarrow \mathcal{A}_p. \tag{B.1.2}
$$

The homomorphism (B.1.2) respects the actions of the group of symmetries Γ and Γ_p respectively. This is summarized in the following commutative diagram:

$$
\begin{array}{ccc}
\Gamma \times \mathcal{A}_\hbar & \longrightarrow & \mathcal{A}_\hbar \\
{\scriptstyle (p,q)} \downarrow & & \downarrow {\scriptstyle q} \\
\Gamma_p \times \mathcal{A}_p & \longrightarrow & \mathcal{A}_p,
\end{array} \tag{B.1.3}
$$

where $p : \Gamma \longrightarrow \Gamma_p$ is the canonical quotient map.

Step 3. Next, we construct an explicit representation of $\mathcal{A}_p$:

$$
\pi_p : \mathcal{A}_p \longrightarrow \mathrm{End}(\mathcal{H}).
$$

---

[15]This statement and proof work more generally for all Planck constants of the form $\hbar = M/N$, with M, N co-prime integers, and N odd. One should replace then the finite Field $\mathbb{F}_p$ with the finite ring $\mathbb{Z}/N\mathbb{Z}$.

Let $V = V_1 \bigoplus V_2$ be a Lagrangian decomposition of V. In our case V is two-dimensional, therefore V_1 and V_2 are linearly independent lines in V. Take $\mathcal{H} = \mathcal{S}(V_1)$ to be the vector space of complex valued functions on V_1. For an element $v \in V$ define:

$$\pi_p(v) = \psi(\tfrac{1}{2}\,\omega(v_1, v_2))L_{v_1}M_{v_2}, \tag{B.1.4}$$

where $v = v_1 + v_2$ is a direct decomposition $v_1 \in V_1$, $v_2 \in V_2$, L_{v_1} is the translation operator defined by v_1:

$$L_{v_1}(f)(x) = f(x + v_1), \quad f \in \mathcal{S}(V_1)$$

and M_{v_2} is a notation for the operator of multiplication by the function $M_{v_2}(x) = \psi(\omega(v_2, x))$. Next, we show that the formulas given in (B.1.4) satisfy the relations (B.1.1) and thus constitute a representation of the algebra $\mathcal{A}_p$. Let $u, v \in V$. We have to show:

$$\pi_p(u + v) = \psi(\tfrac{1}{2}\,\omega(u, v))\pi_p(u)\pi_p(v).$$

Compute:

$$\pi_p(u + v) = \pi_p((u_1 + u_2) + (v_1 + v_2)),$$

where $u = u_1 + u_2$ and $v = v_1 + v_2$ are decompositions of u and v correspondingly.

Then:

$$\pi_p((u_1 + v_1) + (u_2 + v_2)) = \psi(\tfrac{1}{2}\,\omega(u_1 + v_1, u_2 + v_2))L_{u_1+v_1}M_{u_2+v_2}. \tag{B.1.5}$$

This is by definition of π_p (cf. (B.1.4)). Now use the following formulas:

$$\begin{aligned}
L_{u_1+v_1} &= L_{u_1}L_{v_1}, \\
L_{v_1}M_{u_2} &= M_{u_2}(v_1)L_{v_1}
\end{aligned}$$

to obtain that the right-hand side of (B.1.5) is equal to:

$$\psi(\tfrac{1}{2}\,\omega(u_1 + v_1, u_2 + v_2) + \omega(u_2, v_1))L_{u_1}M_{u_2}L_{v_1}M_{v_2}.$$

Now use:

$$\tfrac{1}{2}\,\omega(u_1 + v_1, u_2 + v_2) + \omega(u_2, v_1) = \tfrac{1}{2}\,\omega(u_1 + u_2, v_1 + v_2) + \tfrac{1}{2}\,\omega(u_1, u_2) + \tfrac{1}{2}\,\omega(v_1, v_2),$$

to obtain the result:

$$\psi(\tfrac{1}{2}\,\omega(u, v))\pi_p(u)\pi_p(v),$$

which completes the argument.

As a consequence of constructing π_p we automatically proved that $\mathcal{A}_p$ is non-trivial. It is well known that all linear operators on $\mathcal{S}(V_1)$ are linear combinations of translation operators and multiplication by characters. Therefore, $\pi_p : \mathcal{A}_p \longrightarrow \mathrm{End}(\mathcal{H})$ is surjective. But, $\dim(\mathcal{A}_p) \leq p^2$ therefore π_p is a bijection. This means that $\mathcal{A}_p$ is isomorphic to a matrix algebra $\mathcal{A}_p \simeq M_p(\mathbb{C})$.

Step 4. Completing the proof of existence. The group Γ_p acts on $\mathcal{A}_p$. Therefore, it acts

on the category of its representations. However, $\mathcal{A}_p$ is isomorphic to a matrix algebra, therefore it has unique irreducible representation, up to isomorphism. This is the standard representation being of dimension p. But $\dim(\mathcal{H}) = p$, therefore π_p is an irreducible representation and its isomorphism class is fixed by Γ_p meaning that we have a pair:

$$\begin{aligned}
\pi_p : \mathcal{A}_p &\longrightarrow \operatorname{End}(\mathcal{H}), \\
\rho_p : \Gamma_p &\longrightarrow \operatorname{PGL}(\mathcal{H})
\end{aligned}$$

satisfying the Egorov identity:

$$\rho_p(B)\pi_p(v)\rho_p(B^{-1}) = \pi_p(Bv),$$

where $B \in \Gamma_p$ and $v \in \mathcal{A}_p$.

It is a well known general fact (attributed to I. Schur) that the group Γ_p, where p is an odd prime, has no non-trivial projective representations. This means that ρ_p can be linearized[16] to give:

$$\rho_p : \Gamma_p \longrightarrow \operatorname{GL}(\mathcal{H}).$$

Now take:

$$\begin{aligned}
\mathcal{H}_\hbar &= \mathcal{H}, \\
\pi_\hbar &= \pi_p \circ q, \\
\rho_\hbar &= \rho_p \circ p.
\end{aligned}$$

Because q intertwines the actions of Γ and Γ_p (cf. diagram (B.1.3)) we see that $\pi_\hbar$ and $\rho_\hbar$ are compatible, namely, the Egorov identity is satisfied:

$$\rho_\hbar(B)\pi_\hbar(f)\rho_\hbar(B^{-1}) = \pi_\hbar(f^B),$$

where $B \in \Gamma_p$ and $f \in \mathcal{A}_\hbar$. Here the notation $\pi_\hbar(f^B)$ means to apply any preimage $\overline{B} \in \Gamma$ of $B \in \Gamma_p$ on f. In particular, this implies that the isomorphism class of $\pi_\hbar$ is fixed by Γ. Knowing that such representation $\pi_\hbar$ is unique up to an isomorphism, (Theorem 2.1), our desired object has been obtained. ∎
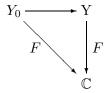
B.2 Proof of Lemma 4.3

Basic set-up: let $(\pi_\hbar, \mathcal{H}_\hbar)$ be a representation of $\mathcal{A}_\hbar$, which is a representative of the unique irreducible class which is fixed by Γ (cf. Theorem 2.1). Let $\rho_\hbar : \Gamma_p \longrightarrow \operatorname{GL}(\mathcal{H}_\hbar)$ be the associated honest representation of the quotient group Γ_p (see Theorem 2.2 and Proof B.1). Recall the notation $Y_0 = \Gamma_p \times \Lambda^*$. We consider the function $F : Y_0 \longrightarrow \mathbb{C}$ defined by the following formula:

$$F(B, \xi) = \operatorname{Tr}(\rho_\hbar(B)\pi_\hbar(\xi)), \tag{B.2.1}$$

---

[16] see Appendix A for an independent proof based on *"The method of canonical Hilbert space"*.

where $\xi \in \Lambda^*$ and $B \in \Gamma_p$. We want to show that F factors through the quotient set $Y = \Gamma_p \times V$:

$$
\begin{array}{ccc}
Y_0 & \longrightarrow & Y \\
& {\scriptstyle F} \searrow & \downarrow {\scriptstyle F} \\
& & \mathbb{C}
\end{array}
$$

The proof is immediate, taking into account the construction given in section B.1. Let π_p be the unique (up to isomorphism) representation of the quotient algebra $\mathcal{A}_p$. As was stated in B.1, $\pi_\hbar$ is isomorphic to $\pi_p \circ q$, where $q : \mathcal{A}_\hbar \longrightarrow \mathcal{A}_p$ is the quotient homomorphism between the algebras. This means that $\pi_\hbar(\xi) = \pi_p(q(\xi))$ depends only on the image $q(\xi) \in V$, and formula (B.2.1) solves the problem. ∎

B.3   Proof of Theorem 4.4

Basic set-up: in this section we use the notations of section B.1 and Appendix A. Set $Y = \mathrm{Sp} \times V$ and let $\alpha : \mathrm{Sp} \times Y \longrightarrow Y$ denote the associated action map. Let $F : Y \longrightarrow \mathbb{C}$ be the function appearing in the statement of Theorem 4.4, i.e., $F(B, v) = \mathrm{Tr}(\rho_p(B)\pi_p(v))$, where $B \in \mathrm{Sp}$ and $v \in V$. We use the notations $\mathbf{V}$, $\mathbf{Sp}$ and $\mathbf{Y}$ to denote the corresponding algebraic varieties. For the convenience of the reader we repeat here the formulation of the theorem:

Theorem B.1 (Geometrization Theorem) *There exists a Weil object $\mathcal{F} \in \mathrm{D}_c^b(\mathbf{Y})$ satisfying the following properties:*

1. (Perversity) The object $\mathcal{F}$ is geometrically irreducible $[\dim(\mathbf{Y})]$-perverse of pure weight $w(\mathcal{F}) = 0$.

2. (Function) The function F is associated to $\mathcal{F}$ via *sheaf-to-function correspondence*:

$$
f^{\mathcal{F}} = F.
$$

3. (Equivariance) For every element $S \in \mathbf{Sp}$ there exists an isomorphism:

$$
\alpha_S^* \mathcal{F} \simeq \mathcal{F}.
$$

4. (Formula) Restricting the sheaf $\mathcal{F}$ on the closed subvariety $\mathbf{T}^\times \times \mathbf{V}$. Using the identifications (4.3.1), (4.3.2), we have an explicit formula:

$$
\mathcal{F}|_{\mathbf{T}^\times \times \mathbf{V}}\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right), \lambda, \mu\right) \simeq \mathscr{L}_{\sigma(a)} \otimes \mathscr{L}_{\psi(\frac{1}{2}\frac{a+1}{a-1}\lambda \cdot \mu)}.
$$

Construction of the sheaf $\mathcal{F}$. We use the notations of Appendix A. Let H be the Heisenberg group. As a set we have H $= $ V $\times \mathbb{F}_q$. The group structure is given by the multiplication rule $(v, \lambda) \cdot (v', \lambda') = (v + v', \lambda + \lambda' + \frac{1}{2}\omega(v, v'))$. We fix a section $s : $ V $\dashrightarrow$ H, $s(v) = (v, 0)$. The group Sp acts by automorphisms on the group H, through its tautological action on the vector space V, i.e., $g \cdot (v, \lambda) = (gv, \lambda)$. We define the semi-direct product G $=$ Sp $\ltimes$ H. We consider the map $(\mathrm{Id}, s) : $ Y $\longrightarrow$ G. We use the notations $\mathbf{H}$ and $\mathbf{G}$ to denote the corresponding algebraic varieties.

Let $\mathcal{K}$ be the Weil representation sheaf (see Theorem A.5). *Define*:

$$\mathcal{F} = \mathrm{Tr}(\mathcal{K}_{|\mathbf{Y}}),$$

where Tr is defined in complete analogue to the operation of taking trace in the set-theoretic framework, that is, we take:

$$\mathcal{F}(g, v) = \int_{x \in \mathbb{A}^1} \mathcal{K}(g, v, x, x),$$

where we use the notation $\int_{x \in \mathbb{A}^1}$ to denote integration with compact support along the x-variable. We prove that the sheaf $\mathcal{F}$ satisfies Properties 1 - 4.

Proof of Property 2. Property 2 follows easily. One should observe that the collection of operators $\{\pi_p(v)\}_{v \in V}$ extends to a representation of the group H, which we will also denote by π_p. Both the representations ρ_p and π_p glue to a single representation $\rho_p \ltimes \pi_p$ of the semi-direct product G. It is a direct verification, that the representation $\rho_p \ltimes \pi_p$ is isomorphic to the representation $\rho \ltimes \pi$ constructed in Appendix A. Hence we can write:

$$f^{\mathcal{F}} = f^{\mathrm{Tr}(\mathcal{K}_{|\mathbf{Y}})} = \mathrm{Tr}(f^{\mathcal{K}_{|\mathbf{Y}}}) = \mathrm{Tr}(K_{|\mathbf{Y}}) = F.$$

In the above equation we use the fact that the operation of taking geometric trace commutes with sheaf-to-function correspondence. This proves Property 2.

Proof of Property 3. Principally follows from the multiplication property of the sheaf $\mathcal{K}$ (Theorem A.5, Property 2). More precisely, using the multiplication property we obtain the following isomorphism:

$$\mathcal{K}_{|S} * \mathcal{K} * \mathcal{K}_{|S^{-1}} \simeq L_S^* R_{S^{-1}}^* \mathcal{K},$$

where L_S, $R_{S^{-1}}$ denotes left multiplication by S and right multiplication by S^{-1} on the group $\mathbf{G}$ respectively. Next, we have:

$$\alpha_S^* \mathcal{F} \simeq \mathrm{Tr}(\alpha_S^* \mathcal{K}_{|\mathbf{Y}}) \simeq \mathrm{Tr}(L_S^* R_{S^{-1}}^* \mathcal{K}_{|\mathbf{Y}}) \simeq \mathrm{Tr}(\mathcal{K}_{|S} * \mathcal{K}_{|\mathbf{Y}} * \mathcal{K}_{|S^{-1}}).$$

Finally, we have the following isomorphisms:

$$\mathrm{Tr}(\mathcal{K}_{|S} * \mathcal{K}_{|\mathbf{Y}} * \mathcal{K}_{|S^{-1}}) \simeq \mathrm{Tr}(\mathcal{K}_{|S^{-1}} * \mathcal{K}_{|S} * \mathcal{K}_{|\mathbf{Y}}),$$

and:

$$\mathcal{K}_{|_{S^{-1}}} * \mathcal{K}_{|_S} \simeq \mathcal{I},$$

where the first isomorphism is the basic property of the trace. Its proof in the geometric setting, is a result of a direct diagram chasing. Structurally, it follows the same lines as in the usual set-theoretic setting. The second isomorphism is a consequence of the multiplication property of $\mathcal{K}$. This completes the proof of Property 3.

Proof of Property 4. This property is directly verified using the explicit formulas appearing in A.3.

Proof of Property 1. We will use the notations $\mathcal{K}_\rho$, and $\mathcal{K}_\pi$ to denote the restriction of the sheaf $\mathcal{K}$ to the subgroups **Sp**, and **H** respectively. We recall that we have the formula $\mathcal{K}_\pi(h, x, y) = \mathscr{L}_{\psi(\frac{1}{2}pq+px+\lambda)} \otimes \delta_{y=x+q}$, where $h = (q, p, \lambda)$. Moreover, it is easy to verify that the sheaf $\mathcal{K}_\rho$ is irreducible $[\dim(\mathbf{Sp}) + 1]$-perverse Weil sheaf of pure weight zero. Finally, we have that:

$$\mathcal{K} \simeq \mathcal{K}_\rho * \mathcal{K}_\pi. \tag{B.3.1}$$

Let $h = (q, p, 0)$. Now write:

$$
\begin{aligned}
\mathcal{F}(g, h) & \simeq \text{Tr}(\mathcal{K}(g, h)) \\
& \simeq \int_{x \in \mathbb{A}^1} \mathcal{K}_\rho(g, x, x - q) \otimes \mathscr{L}_{\psi(-\frac{pq}{2}+px)} \\
& \simeq \int_{x \in \mathbb{A}^1} \mathcal{K}_\rho(g, x, x - q) \otimes \mathscr{L}_{\psi(\frac{1}{2}p(\frac{x}{2}-q))} \\
& \simeq \int_{x \in \mathbb{A}^1} \mathcal{K}_\rho(g, \frac{x + q}{2}, \frac{x - q}{2}) \otimes \mathscr{L}_{\psi(\frac{1}{2}px)}.
\end{aligned}
$$

Where the first isomorphism is a direct consequence of the formulas of the sheaves involved and isomorphism (B.3.1). The third isomorphism is a change of variables $(q, \frac{x}{2} - q) \to (q, x)$. Consider the isomorphism $\beta : \mathbf{Sp} \times \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbf{Sp} \times \mathbb{A}^1 \times \mathbb{A}^1$, defined by $\beta(g, q, x) = (g, \frac{x+q}{2}, \frac{x-q}{2})$. The last term in (B.3.2) is equivalent to taking (non-normalized) fiber-wise Fourier transform of $\beta^* \mathcal{K}_\rho$ considered as a sheaf on the line bundle $(\mathbf{Sp} \times \mathbb{A}^1) \times \mathbb{A}^1 \to (\mathbf{Sp} \times \mathbb{A}^1)$. This implies using the theory of the ℓ-adic Fourier transform (cf. [KL]) that $\mathcal{F}$ is geometrically irreducible $[\dim(\mathbf{Y})]$-perverse of pure weight zero. This concludes the proof of Property 1, and of Theorem 4.4. ∎

B.4 Computations for the Vanishing Lemma (Lemma 4.6)

In the computations we use some finer technical tools from the theory of ℓ-adic cohomology. The interested reader can find a systematic study of this material in [K, KW, L, BBD, BL].

We identify the standard torus $\mathbf{T} \subset \mathbf{SL_2}$ with the group $\mathbb{G}_m$. Fix a non-trivial character sheaf[17] $\mathscr{L}_\chi$ on $\mathbb{G}_m$. Denote by $\mathscr{L}_\psi$ a non-trivial character sheaf on $\mathbb{G}_a$. Fix $\lambda, \mu \in \mathbb{A}^1$ with $\lambda \cdot \mu \neq 0$. Consider the variety $\mathbf{X} = \mathbb{G}_m - \{1\}$, the sheaf:

$$\mathcal{E} = \mathscr{L}_{\psi(\frac{1}{2}\frac{a+1}{a-1}\lambda\cdot\mu)} \otimes \mathscr{L}_\chi, \tag{B.4.1}$$

on $\mathbf{X}$ and the canonical projection $pr : \mathbf{X} \longrightarrow pt$. Note that $\mathcal{E}$ is a non-trivial 1-dimensional local system on $\mathbf{X}$. The proof of the Lemma will be given in several steps:

Step 1. Vanishing. We want to show that $\mathrm{H}^i(pr_!\mathcal{E}) = 0$ for $i = 0, 2$.

By definition:

$$\mathrm{H}^0(pr_!\mathcal{E}) = \Gamma(\mathbf{Y}, j_!\mathcal{E}),$$

where $j : \mathbf{X} \hookrightarrow \mathbf{Y}$ is the imbedding of $\mathbf{X}$ into a compact curve $\mathbf{Y}$. The statement follows since:

$$\Gamma(\mathbf{Y}, j_!\mathcal{E}) = \mathrm{Hom}(\overline{\mathbb{Q}}_\ell, j_!\mathcal{E})$$

and it is easy to see that any non-trivial morphism $\overline{\mathbb{Q}}_\ell \to j_!\mathcal{E}$ should be an isomorphism, hence $\mathrm{Hom}(\overline{\mathbb{Q}}_\ell, j_!\mathcal{E}) = 0$.

For the second cohomology we have:

$$\mathrm{H}^2(pr_!\mathcal{E}) = \mathrm{H}^{-2}(Dpr_!\mathcal{E})^* = \mathrm{H}^{-2}(pr_* D\mathcal{E})^* = \Gamma(\mathbf{X}, D\mathcal{E}[-2])^*,$$

where D denotes the Verdier duality functor[18] and $[-2]$ means translation functor. The first equality follows from the definition of D, the second equality is the Poincaré duality and the third equality easily follows from the definitions. Again, since the sheaf $D\mathcal{E}[-2]$ is a non-trivial 1-dimensional local system on $\mathbf{X}$ then:

$$\Gamma(\mathbf{X}, D\mathcal{E}[-2]) = \mathrm{Hom}(\overline{\mathbb{Q}}_\ell, D\mathcal{E}[-2]) = 0.$$

Step 2. Dimension. We claim that $\dim \mathrm{H}^1(pr_!\mathcal{E}) = 2$.

The (topological) *Euler characteristic* $\chi(pr_!\mathcal{E})$ of the sheaf $pr_!\mathcal{E}$ is the integer defined by the formula:

$$\chi(pr_!\mathcal{E}) = \sum_i (-1)^i \dim \mathrm{H}^i(pr_!\mathcal{E}).$$

---

[17]That is, a 1-dimensional local system on $\mathbb{G}_m$ that satisfies the property $m^*\mathscr{L}_\chi \simeq \mathscr{L}_\chi \boxtimes \mathscr{L}_\chi$, where $m : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ is the multiplication morphism.

[18]In the vector bundle interpretation, one might think on D as the operation of taking the dual vector bundle with the dual connection.

Hence from the vanishing of cohomologies (Step 1) we deduce:

Substep 2.1. It is enough to show that $\chi(pr_!\mathcal{E}) = -2$.

The actual computation of the Euler characteristic $\chi(pr_!\mathcal{E})$ is done using the *Ogg-Shafarevich-Grothendieck formula* [D3]:

$$\mathrm{rk}(\mathcal{E}) \cdot \chi(pr_!\overline{\mathbb{Q}}_\ell) - \chi(pr_!\mathcal{E}) = \sum_{y \in \mathbf{Y} \smallsetminus \mathbf{X}} \mathrm{Swan}_y(\mathcal{E}). \tag{B.4.2}$$

Here $\mathrm{rk}(\mathcal{E}) = 1$ is the rank of the sheaf $\mathcal{E}$, $\overline{\mathbb{Q}}_\ell$ denotes the constant sheaf on $\mathbf{X}$, and $\mathbf{Y}$ is some compact curve containing $\mathbf{X}$. In other words, this formula expresses the difference of $\mathrm{rk}(\mathcal{E}) \cdot \chi(pr_!\overline{\mathbb{Q}}_\ell)$ from $\chi(pr_!\mathcal{E})$ as a sum of local contributions, called *Swan conductors*. We will not to give here the formal definition of the Swan conductor (see [K, KL, L]), but instead we will formulate some of the properties and known results that are needed for our calculations.

Next, we take $\mathbf{Y} = \mathbb{P}^1$. Having that, $\chi(pr_!\overline{\mathbb{Q}}_\ell) = -1$, and using formula (B.4.2) we get:

Substep 2.2. It is enough to show that: $\mathrm{Swan}_0(\mathcal{E}) + \mathrm{Swan}_1(\mathcal{E}) + \mathrm{Swan}_\infty(\mathcal{E}) = 1$.

We would like now to understand the affect of tensor product on the Swan conductor. Choose a point $y \in \mathbf{Y} \smallsetminus \mathbf{X}$, and let $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{D}_c^b(\mathbf{X})$ be two sheaves with $\mathcal{L}_1$ being *lisse* (smooth) in a neighborhood of y. We have:

$$\mathrm{Swan}_y(\mathcal{L}_1 \otimes \mathcal{L}_2) = \mathrm{rk}(\mathcal{L}_1) \cdot \mathrm{Swan}_y(\mathcal{L}_2). \tag{B.4.3}$$

In particular, using property (B.4.3), and the explicit formula (B.4.1) of the sheaf $\mathcal{E}$, we deduce that:

$$\begin{aligned}
\mathrm{Swan}_1(\mathcal{E}) &= \mathrm{Swan}_\infty(\mathscr{L}_\psi), \\
\mathrm{Swan}_\infty(\mathcal{E}) &= \mathrm{Swan}_\infty(\mathscr{L}_\chi), \\
\mathrm{Swan}_0(\mathcal{E}) &= \mathrm{Swan}_0(\mathscr{L}_\chi).
\end{aligned}$$

Substep 2.3. We have $\mathrm{Swan}_\infty(\mathscr{L}_\psi) = 1$, and $\mathrm{Swan}_\infty(\mathscr{L}_\chi) + \mathrm{Swan}_0(\mathscr{L}_\chi) = 0$.

Applying the Ogg-Shafarevich-Grothendieck formula to the Artin-Schreier sheaf $\mathscr{L}_\psi$ on $\mathbb{A}^1$ and the projection $pr : \mathbb{A}^1 \longrightarrow pt$ we find that:

$$\mathrm{Swan}_\infty(\mathscr{L}_\psi) = \chi(pr_!\overline{\mathbb{Q}}_\ell) - \chi(pr_!\mathscr{L}_\psi) = 1 - 0 = 1. \tag{B.4.4}$$

Finally, we apply the formula (B.4.2) to the sheaf $\mathscr{L}_\chi$ on $\mathbb{G}_m$ and the projection $pr : \mathbb{G}_m \longrightarrow pt$ and conclude:

$$\mathrm{Swan}_\infty(\mathscr{L}_\chi) + \mathrm{Swan}_0(\mathscr{L}_\chi) = \chi(pr_!\overline{\mathbb{Q}}_\ell) - \chi(pr_!\mathscr{L}_\chi) = 0 - 0 = 0. \tag{B.4.5}$$

Note that, in (B.4.4) and (B.4.5) we use the fact that $pr_!\mathscr{L}_\psi$ and $pr_!\mathscr{L}_\chi$ are the $0-$objects in $\mathrm{D}^b_\mathrm{c}(pt)$.

This completes the computations of the Vanishing Lemma. ∎

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